Sunflowers and Ramsey Problems for Restricted Intersections

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Thm (Erdős–Ko–Rado 1938). $\mathcal{F} \subset \binom{[n]}{k}$, $n \geq 2k$

If $|F \cap F'| \geq 1$ $\forall F \neq F' \in \mathcal{F}$ $\Rightarrow$ $|\mathcal{F}| \leq \binom{n-1}{k-1}$

$L = \{1, 2, \cdots, k-1\}$ $\quad |L| = k-1$

Set systems with restricted intersections.

Given k and $L \subset \{0, \cdots, k-1\}$. $\mathcal{F} \subset \binom{[n]}{k}$ is an <u>(n, k, L)-system</u> if $|F \cap F'| \in L$ $\forall F \neq F' \in \mathcal{F}$

Q: $|\mathcal{F}| \leq \boxed{?}$

central in extremal set theory

applications in Ramsey graphs, discrete geometry, coding theory...

Thm (Ray-Chaudhuri – Wilson 1975). $\mathcal{F}$ ... (n,k,L)-system
$\Rightarrow |\mathcal{F}| \le \binom{n}{|L|}$

Thm (Frankl – Wilson 1981). p ... prime. $r_1 \cdots r_s$ distinct residues mod p.
$k \bmod p \notin \{r_1 \cdots r_s\}$. $\mathcal{F} \subset \binom{[n]}{k}$. $|F \cap F'| \bmod p \in \{r_1, \cdots r_s\}$ $\forall F \neq F' \in \mathcal{F}$
$\Rightarrow |\mathcal{F}| \le \binom{n}{s}$.

$L = \{0 \le l < k : l \bmod p \in \{r_1 \cdots r_s\}\}$

$\mathcal{F} \subset \binom{[n]}{k}$ $\quad L \subset \{0, 1, \cdots, k-1\}$

build $G_{\mathcal{F}}$ $\begin{cases} V(G_{\mathcal{F}}) = \mathcal{F} \\ F \sim F' \text{ iff } |F \cap F'| \in L \end{cases}$

- L-clique ($F_1 \cdots F_m$ s.t. $|F_i \cap F_j| \in L$) ... (n,k,L)-system
- L-avoiding family ($F_1 \cdots F_m$ s.t. $|F_i \cap F_j| \notin L$) ... $(n,k,\{0,\cdots,k-1\} \setminus L)$-system

Q: if $G_{\mathcal F}$ has no ^{L}clique of $m+1$

$\Longrightarrow$ how large is $\alpha(G_{\mathcal F})$?

$\Longleftarrow$) If $\mathcal F$ has no L-clique of size $m+1$

~~If $\mathcal F$ has no L-clique of size $m+1$ $\Rightarrow$ how large can an L-avoiding $\mathcal F' \subset \mathcal F$?~~

$\Rightarrow$ how large can an L-avoiding $\mathcal F' \subset \mathcal F$?

— Apply Ramsey $\quad \omega(G_{\mathcal F}) \le m \Rightarrow \alpha(G_{\mathcal F}) \gtrsim |\mathcal F|^{\frac{1}{m}}$

— $\dfrac{|\mathcal F'|}{|\mathcal F|} \ge \boxed{?}$

Eg: $\mathcal F \subset \binom{[n]}{2k}$ $\qquad L = ODD = \{1, 3, 5, 7, \cdots, 2k-1\}$

If $\mathcal F$ has no ODD-clique of size $m+1$

$$\Longrightarrow \exists \mathcal F' \subset \mathcal F \text{ s.t. } \begin{cases} |F \cap F'| \text{ is even} \\ |\mathcal F'| \ge ? \end{cases}$$

- Used in quantum computing

Take $\mathcal F = \binom{[n]}{2k} \simeq n^{2k}$

$|\mathcal F|$

$|F_i| \bmod 2 = 0$

— ODD-clique. $F_1 \cdots F_m$ s.t. $|F_i \cap F_j| \bmod 2 = 1$

$\Rightarrow$ Frankl–Wilson $\Rightarrow m \leq n$

— $\mathcal{F}' \subset \mathcal{F}$. $|F \cap F'| \bmod 2 = 0$

$\Rightarrow$ Ray-Chaudhuri–Wilson : $|\mathcal{F}'| \leq \binom{n}{k}$

$\max |\mathcal{F}'| = \binom{n/2}{k}$ $\qquad |F \cap F'| \in \{0, 2, 4, \dots, 2k-2\}$

$$|\mathcal{F}'| \lesssim n^k \approx \frac{|\mathcal{F}|}{n^k} \approx \frac{|\mathcal{F}|}{m^k}$$

Thm (Janzer–J.–Sudakov–Wu 25+)

$\mathcal{F} \subset \binom{[n]}{2k}$. no ODD-clique of $m+1$

$\Rightarrow \exists\, \mathcal{F}' \subset \mathcal{F}$: $\begin{cases} \text{ODD-avoiding} \\ |\mathcal{F}'| = \Omega_k(m^{-k}|\mathcal{F}|) \end{cases}$

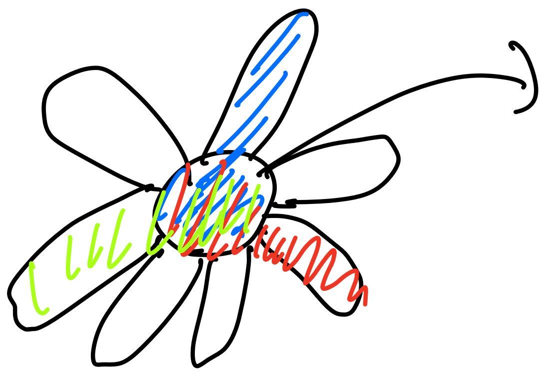
- $m^{-k}|\mathcal{F}|$ is tight upto a constant dependent on k.

$\binom{[n]}{k}$ $\quad L \subset \{0, \dots, k-1\}$

Q: If $\mathcal{F}$ has no L-clique of size $m+1$

$\Rightarrow$ how large can an L-avoiding $\mathcal{F}' \subset \mathcal{F}$?

L-clique



Def (sunflower): a collection k-element sets

$A_1 \dots A_m$ forms a sunflower if

$$A_i \cap A_j = A_{i'} \cap A_{j'} \quad \forall i \neq j, \ \forall i' \neq j'$$

kernel : $A_1 \cap A_2 \cap \dots \cap A_m$

petal : each A_i is a petal

Def: L-sunflower : $|\text{kernel}| \in L$

Obs: L-sunflower $\Rightarrow$ L-clique

$\{\ell\}$-clique $\Longrightarrow$ $\{\ell\}$-sunflower

$\{\ell\}$-clique has size $\geq k^2-k+2$

sunflowers are important tests

sunflower conjecture (Erdős-Rado):

$\mathcal{A}$ $\begin{cases} \text{a collection of } k\text{-element sets} \\ \text{no sunflower of } m \text{ petals} \end{cases}$

$\Rightarrow |\mathcal{A}| \leq (f(m))^k$.

Q: (1) If $\mathcal{F}$ has no L-clique of size $m+1$

$\Rightarrow$ how large can an L-avoiding $\mathcal{F}' \subset \mathcal{F}$?

(2)

If $\mathcal{F}$ has no L-sunflower of size $m+1$

$\Rightarrow$ how large can an L-avoiding $\mathcal{F}' \subset \mathcal{F}$?

- (1) = (2) if $L = \{\ell\}$ + $m \geq k^2-k+2$
- Are these two questions the same?

No! We solved (2)

and proved something for the modular setting of (1)

$\boxed{\text{Focus on } L = \{\ell\}.\ \ell > 0}$

Q (Erdős–Sós) $\mathcal{F} \subset \binom{[n]}{k}$ is $(n, k, \{0, \dots, k-1\} \setminus \{\ell\})$-system, $|F \cap F'| \neq \ell$

$$\Rightarrow |\mathcal{F}| \leq ES(n,k,\ell)$$

Thm (Frankl–Füredi 85) $ES(n,k,\ell)$

$$= O_k(n^{\max(\ell, k-\ell-1)})$$

Q (Duke – Erdős) $\mathcal{F} \subset \binom{[n]}{k}$. no l-sunflower of $m+1$ petals

$$\Rightarrow |\mathcal{F}| \leq DE(n,k,l,m)$$

- $DE(n,k,l,1) = ES(n,k,l)$

Thm (Bradač – Bucić – Sudakov 21)

$$DE(n,k,l,m) = \Theta_k\left(n^{\max(l,k-l-1)} \cdot m^{\min(k-l,l+1)}\right)$$

取 $|\mathcal{F}| = DE(n,k,l,m)$. no l-sunflower of $m+1$ petals

$$\mathcal{F} \overset{(2)}{\Longrightarrow} \mathcal{F}' \subset \mathcal{F} \text{ s.t. } |F \cap F'| \neq l \quad \forall F \neq F' \in \mathcal{F}'$$

$$\frac{|\mathcal{F}'|}{|\mathcal{F}|} \geq m^{-(k-l)}.$$

Using $|\mathcal{F}'| \leq ES(n,k,l)$,

$$\Rightarrow \quad \frac{ES(n,k,l)}{DE(n,k,l,m)} \gtrsim_k m^{-(k-l)}$$

$\Rightarrow \quad PE(n,k,l,m) \lesssim_k m^{k-l}\, ES(mk,l)$

$l \geq \frac{k-1}{2} \to$ recovers BBS

Thm (Janzer - J. - Sudakov - Wu 25+)

$\mathcal{F}$ has no l-sunflower of $m+1$ petals

$\Rightarrow \exists\, \mathcal{F}' \subset \mathcal{F}$ $\begin{cases} |F \cap F'| \neq l \quad (l\text{-avoiding}) \\ \frac{|\mathcal{F}'|}{|\mathcal{F}|} \geq c_k \cdot m^{-(k-l)}. \end{cases}$

In addition, $m^{-(k-l)}$ is tight.

A proof that gives m^{-k}

$\mathcal{F} \subset \binom{[n]}{k}$. no l-sunflower of $m+1$ petals.

Consider ... $A \subset [n]$ ... and look at

Consider any $A \subset [m]$ of size ℓ. and look at

ℓ
$=$
$|A|$

all $F \in \mathcal{F}$ with $A \subset F$

$\Rightarrow F \setminus A$ don't form matching of size $m+1$

$\Rightarrow \exists\, \varphi(A) \subset [n] \setminus A$

$|\varphi(A)| \le m \cdot k$

Any $F \in \mathcal{F}$ with $A \subset F$ must have $\varphi(A) \cap F \neq \emptyset$

• good if $|\varphi(A)| \le 1 \quad \forall A$

▷ $|\varphi(A)| = 0$ ...... no $A \subset F \in \mathcal{F}$

▷ $|\varphi(A)| = 1$ ...... all $A \subset F \in \mathcal{F}$ must contain v.

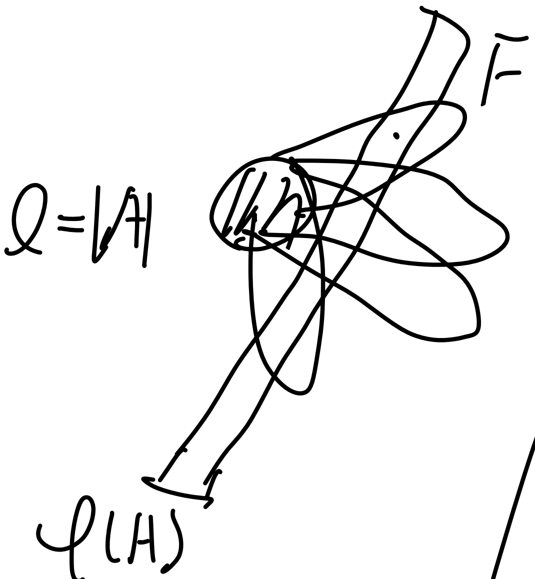
$\varphi(A) = \{v\}$

colour certificate

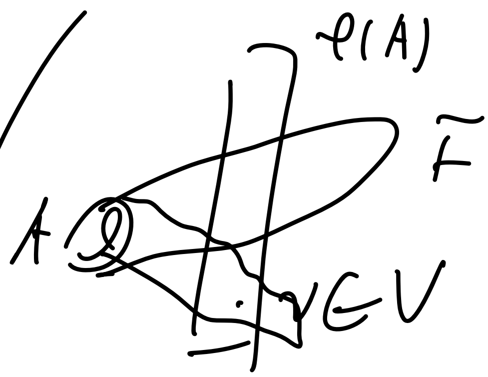
Idea: sample each $v \in [n]$ w.p. $p \approx \frac{\alpha_k}{m}$ to get $V = \{$these $v\}$.

$\mathbb{E}\,|\varphi(A)\cap V| \ll 1$

$$\mathcal{F}' = \{F\in\mathcal{F} : F\subset V\}$$



$$\#\{(F,A,v) : F\in\mathcal{F}',\ A\subset F \text{ of size } \ell,\ v\in\varphi(A)\setminus F,\ v\in V\}$$



$$\mathbb{E}(\quad) \le \mathbb{E}|\mathcal{F}'|/2$$

$$\mathbb{E}|\mathcal{F}'| = |\mathcal{F}|\cdot p^k = \alpha_k^d\cdot \frac{|\mathcal{F}|}{m^k}.$$

Open (1) k-dependency

(2) Q(1) no k-clique

$\Rightarrow L$-avoiding $\mathcal{F}' \subset \mathcal{F}$

$p \qquad L = \{r_1, \ldots, r_s\} \% p$

$s = p-1 \qquad \frac{|\mathcal{F}'|}{|\mathcal{F}|}$ ✓.

$0 < s < p-1 \qquad ?$