Minimum Degree Removal Lemma Thresholds

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Joint work with Lior Gishboliner and Benny Sudakov

The graph removal lemma

Theorem (Ruzsa-Szemerédi '78)

If G contains εn^2 edge-disjoint copies of H, then G contains $\delta n^{v(H)}$ copies of H, where $\delta = \delta_H(\varepsilon) > 0$.

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- One of the first applications of the Szemerédi regularity lemma.
- Proves Roth's theorem: every $S \subseteq \{1, 2, \ldots, n\}$ with no arithmetic progression of length 3 satisfies $|S| = o(n)$.
- Connections to property testing:
  Either G is ε-close to being H-free or $G[S]$ contains H (w.h.p.) for a random $S \subseteq V(G)$ of size $\delta_H^{-1}(\varepsilon)$.

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- Why linear minimum degree?

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Conjecture (Erdős-Simonovits '73)

If G is K_3-free and $\delta(G) > \frac{n}{3}$ then $\chi(G)$ is bounded.

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If G is K_3-free and $\delta(G) > \frac{n}{3}$ then $\chi(G) \leq 4$.

- $\frac{n}{3}$ is the threshold on $\delta(G)$ such that $\chi(G) < \infty$ for all K_3-free G.

The chromatic threshold

Theorem (Allen-Böttcher-Griffiths-Kohayakawa-Morris '13)

The chromatic threshold for H is $\delta_\chi(H) \in \{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\}$, where $\chi(H) = r$. In other words,

- *any H-free G with $\delta(G) > (\delta_\chi(H) + \varepsilon)n$ has $\chi(G) = O_{H,\varepsilon}(1)$;*
- *there exists G on n vertices with $\delta(G) = (\delta_\chi(H) - o(1))n$ with $\chi(G) \to \infty$.*

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- *there exists G on n vertices with* $\delta(G) = (\delta_\chi(H) - o(1))n$ *with* $\chi(G) \to \infty$.

- Among all r-chromatic H's, there are only *three* possible values for the chromatic thresholds.

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Definition: The homomorphism threshold $\delta_{\text{hom}}(H)$ is the infimum $\gamma > 0$ such that if G is H-free and $\delta(G) \geq \gamma n$ then G is homomorphic to an H-free graph on $C(\gamma)$ vertices.

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$\implies \delta_{\text{hom}}(K_3) = \frac{1}{3}$.

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$\delta_{hom}(\{C_3, C_5 \ldots, C_{2k+1}\}) = \frac{1}{2k+1}$ *and* $\delta_{hom}(C_{2k+1}) \leq \frac{1}{2k+1}$.

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Theorem (Sankar '22+): $\delta_{\text{hom}}(C_5) > 0$.

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- *There are graphs G with $\delta(G) = (\frac{2r-5}{2r-3} - \alpha)n$ and εn^2 edge-disjoint copies of K_r, but only $\varepsilon^{\Omega(\log 1/\varepsilon)} n^r$ copies of K_r.*

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$\implies \delta_{\text{lin-rem}}(K_r) = \delta_{\text{poly-rem}}(K_r) = \frac{2r-5}{2r-3}$.

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- Do $\delta_{\text{poly-rem}}(H)$, $\delta_{\text{lin-rem}}(H)$ receive finitely or infinitely many values on r-chromatic graphs H?
- Is there a relation between the removal thresholds $\delta_{\text{poly-rem}}(H)$, $\delta_{\text{lin-rem}}(H)$ and $\delta_{\chi}(H)$, $\delta_{\text{hom}}(H)$?

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Corollary

$\delta_{\textit{poly-rem}}(H)$ receives infinitely many values on 3-chromatic H.

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If H is 3-chromatic,

$$\delta_{\textit{lin-rem}}(H) = \begin{cases} \frac{1}{2} & H \text{ has no critical edge,} \\ \frac{1}{3} & H \text{ has a critical edge and contains a triangle,} \\ \frac{1}{4} & H \text{ has a critical edge but no triangle.} \end{cases}$$

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Corollary

$\delta_{lin\text{-}rem}(H)$ receives 3 different values on 3-chromatic H.

Reduction from $\delta_{\text{poly-rem}}(H)$ to $\delta_{\text{hom}}(H)$

Theorem (Gishboliner, J., Sudakov)

$$\delta_{\textit{poly-rem}}(H) \leq \delta_{\textit{hom}}(\mathcal{I}_H) \triangleq \delta_H,$$

where $\mathcal{I}_H = \{H' : H \to H', H' \text{ is minimal}\}$.

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- $\exists F$, which is $\mathcal{I}_H$ free and $v(F) = O_\alpha(1)$, such that: $\tilde{G}$ is $\mathcal{I}_H$-*free* and $\delta(\tilde{G}) > (\delta_H + \alpha)v(\tilde{G}) \implies \tilde{G} \to F$.

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- G is ε-far from being H-free. $\Rightarrow$ G is ε-far from $G \to F$.

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<u>Goal</u>: if G has $\delta(G) > (\delta_H + \alpha)n$ and has $> \varepsilon n^2$ edge-disjoint H, then G contains $\varepsilon^{O_{H,\alpha}(1)} n^{v(H)}$ copies of H.

- $v(F) = O_\alpha(1)$ and G is ε-far from $G \to F$.
- Take a uniform sample $S \subset V(G)$ of size $q = \text{poly}(H, 1/\varepsilon)$.
  - $\delta(G[S]) > (\delta_H + \alpha)q$ with probability $> 1/2$.
  - By [Nakar-Ron], $G[S] \not\to F$ with probability $> 1/2$.
  - $H' \preceq G[S]$ for some $H' \in \mathcal{I}_H$, i.e. $H \to H'$.
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  - H is a subgraph of some blow-up of H'.

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Linear removal lemma threshold when $\chi(H) = 3$

Theorem (Gishboliner, J., Sudakov)

If H is 3-chromatic,

$$\delta_{\text{lin-rem}}(H) = \begin{cases} \frac{1}{2} & H \text{ has no critical edge,} \\ \frac{1}{3} & H \text{ has a critical edge and contains a triangle,} \\ \frac{1}{4} & H \text{ has a critical edge but no triangle.} \end{cases}$$

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- Interesting case: when H has a critical edge but no triangle.

Linear removal lemma threshold when $\chi(H) = 3$

Goal: $\delta_{\text{lin-rem}}(H) \leq \frac{1}{4}$ when H has a critical edge but no triangle.

- How does H look like?

Linear removal lemma threshold when $\chi(H) = 3$

Goal: $\delta_{\text{lin-rem}}(H) \leq \frac{1}{4}$ when H has a critical edge but no triangle.

- How does H look like?
- H is a bipartite graph plus one edge inside one part.

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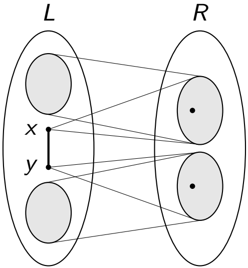

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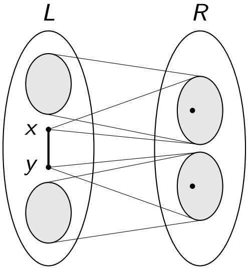

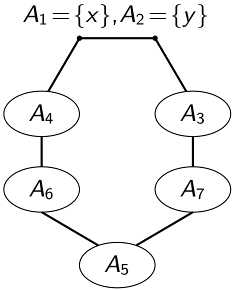

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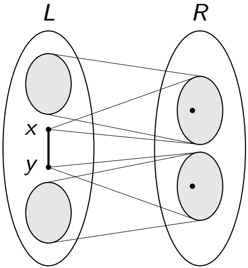

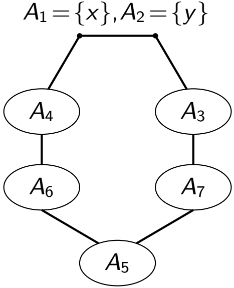

- $H \to C_{2k+1}$ for some $k \geq 2$ with $A_1 = \{x\}, A_2 = \{y\}$.

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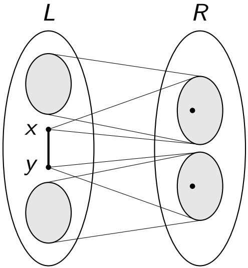

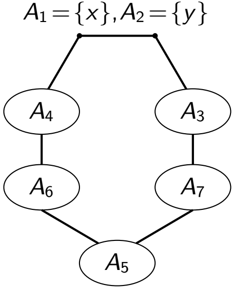

- $H \to C_{2k+1}$ for some $k \geq 2$ with $A_1 = \{x\}, A_2 = \{y\}$.
- Consider $H = C_5$.

Linear removal lemma threshold for C_5

Goal: If G has $\delta(G) > (\frac{1}{4} + \alpha)n$, $> \varepsilon n^2$ edge-disjoint C_5, then G contains $\Omega_\alpha(\varepsilon n^5)$ copies of C_5.

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- If G contains $> \varepsilon^{0.1} n^2$ edge-disjoint C_3 or C_5.
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- Assume G contains $< \varepsilon^{0.1} n^2$ edge-disjoint C_3 or C_5.

An "ideal" proof for C_5

Goal: If G has $\delta(G) > (\frac{1}{4} + \alpha)n$, $> \varepsilon n^2$ edge-disjoint C_5 and $< \varepsilon^{0.1} n^2$ edge-disjoint C_3 or C_5, then G contains $\Omega_\alpha(\varepsilon n^5)$ copies of C_5.

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- Let E_c be the set of edges in these edge-disjoint C_3 and C_5.
  - $|E_c| < 5\varepsilon^{0.1} n^2 < \frac{\alpha^2}{100} n^2$.

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  - "Ideally", $\delta(G') \geq \delta(G) - \frac{\alpha^2}{100} n^2/n > (\frac{1}{4} + \frac{\alpha}{2})n$.
  - "Ideally", G' is bipartite (large degree implies short odd cycle).

An "ideal" proof for C_5

Goal: If G has $> \varepsilon n^2$ edge-disjoint C_5 and G' has $\delta(G') > (\frac{1}{4} + \frac{\alpha}{2})n$ and is bipartite with bipartition $L \sqcup R$, then G contains $\Omega_\alpha(\varepsilon n^5)$ copies of C_5.

- Each C_5 in G contains edge $ab \in L$ or $ab \in R$.

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- Case (a): $\deg_{G'}(a, b) > \frac{\alpha n}{2}$.

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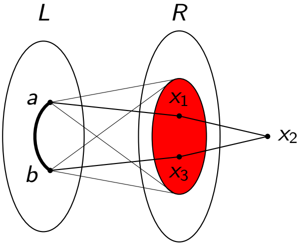

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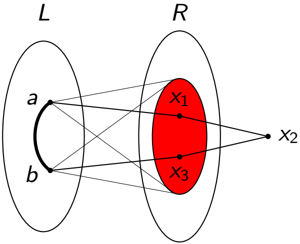

- $\Omega_\alpha(n^3)$ paths (x_1, x_2, x_3) with x_1, x_3 red because $\delta(G) > \frac{1}{4}n$.
- $\Omega_\alpha(n^3)$ copies of C_5's of form (a, x_1, x_2, x_3, b).

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Goal: If G has $> \varepsilon n^2$ edge-disjoint C_5 and G' has $\delta(G') > (\frac{1}{4} + \frac{\alpha}{2})n$ and is bipartite with bipartition $L \sqcup R$, then G contains $\Omega_\alpha(\varepsilon n^5)$ copies of C_5.

- Pick $ab \in L$ (or $ab \in R$).
- Case (b): $\deg_{G'}(a, b) \leq \frac{\alpha n}{2}$.

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- Pick $ab \in L$ (or $ab \in R$).
- Case (b): $\deg_{G'}(a, b) \leq \frac{\alpha n}{2}$.
  - $|R| \geq 2\delta(G') - \deg_{G'}(a, b) > \frac{n}{2}$ and $|L| < \frac{n}{2}$.

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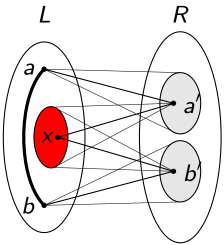

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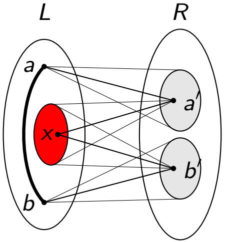

  - $\deg_{G'}(a') + \deg_{G'}(b') > (\frac{1}{2} + \alpha)n > |L| + \alpha n$.

An "ideal" proof for C_5

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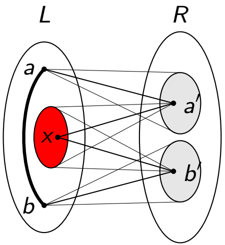

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  - $\Omega_\alpha(n^3)$ C_5's of form (a, a', x, b', b).

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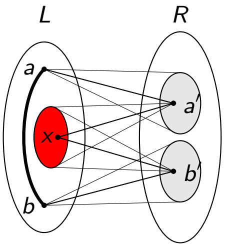

  - $\deg_{G'}(a') + \deg_{G'}(b') > (\frac{1}{2} + \alpha)n > |L| + \alpha n$.
  - $\Omega_\alpha(n^3)$ C_5's of form (a, a', x, b', b).
- $\varepsilon n^2 \cdot \Omega_\alpha(n^3) = \Omega_\alpha(\varepsilon n^5)$ copies of C_5 in total.

The actual proof for $\delta_{\text{lin-rem}}(C_5)$

Goal: If G has $\delta(G) > (\frac{1}{4} + \alpha)n$, $> \varepsilon n^2$ edge-disjoint C_5 and $< \varepsilon^{0.1} n^2$ edge-disjoint C_3 or C_5, then G contains $\Omega_\alpha(\varepsilon n^5)$ C_5.

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  - $|S| \leq |E_c| / \frac{\alpha n}{10} = \frac{\alpha}{10} n$.
- Let $G' := G \setminus E_c \setminus S$.

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- Let $G' := G \setminus E_c \setminus S$.
  - $v(G') \approx v(G)$ and $\delta(G') > (\frac{1}{4} + \frac{\alpha}{2}) v(G')$.

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  - $|S| \leq |E_c| / \frac{\alpha n}{10} = \frac{\alpha}{10} n$.
- Let $G' := G \setminus E_c \setminus S$.
  - $v(G') \approx v(G)$ and $\delta(G') > (\frac{1}{4} + \frac{\alpha}{2}) v(G')$.
  - $oddgirth(G') \geq 7$.

The actual proof for $\delta_{\text{lin-rem}}(C_5)$

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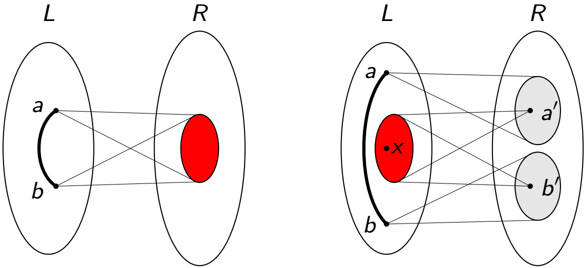
Linear removal lemma threshold for C_5

Case 1: at least $\frac{\varepsilon n^2}{2}$ edges are of type I.

- Same as in the "ideal" proof: pick ab of type I.



Linear removal lemma threshold for C_5

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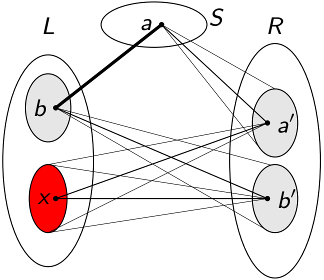

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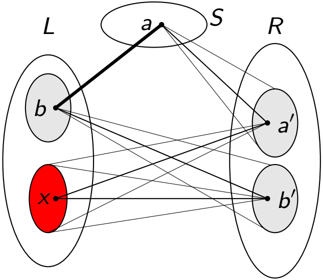

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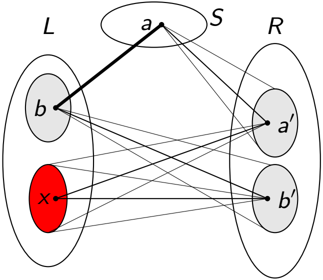

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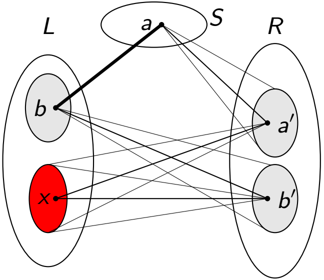

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- $\frac{\varepsilon n^2}{2} \cdot \Omega_\alpha(n^3) = \Omega_\alpha(\varepsilon n^5)$ copies of C_5 in case 2.

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The End

Questions? Comments?