Algebraic aspects of the polynomial Littlewood–Offord problem

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- What can we say when $t = O(\sqrt{n})$?
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  - Gaussian approximation: $\mathcal{N}(0, n)$.
- Can we say something about $\mathbb{P}\left[\sum_i a_i \xi_i = t\right]$ in general?

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Theorem (Erdős–Littlewood–Offord, 1945)

Let $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$. *Let* $\xi_1, \ldots, \xi_n \in \{-1, 1\}$ *be i.i.d. Rademacher r.v.'s. Then,*
$\sup_t \mathbb{P}\left[\sum_i a_i \xi_i = t\right] \leq \binom{n}{\lfloor n/2 \rfloor} / 2^n = O(n^{-1/2})$.

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- State-of-the-art: $\mathbb{P}[\det(A) = 0] = 2^{-n+o(n)}$ by Tikhomirov.

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A combinatorial approach:

Theorem (Kwan and Sauermann 2023+)

If f is a quadratic polynomial with $\Omega(n^2)$ monomials,
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An approach via Gaussian approximation:

Theorem (Kane, Meka-Nguyen-Vu, 2016)

If f is a degree-d polynomial with $\Omega(n^d)$ monomials,
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Suppose f has degree d. Then

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If $q \in \mathbb{Z}[x_1, \ldots, x_m]$ *is irreducible,* $\deg(q) = 2, \operatorname{rank}(q) \geq 2$, *then the number of roots in* $\{-B, -B+1, \ldots, B\}^m$ *is* $O(B^{m-2+o(1)})$.

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- A very hard open problem to understand $\deg(q) \ge 3$!

Our results – quadratic polynomials

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Suppose f is a quadratic polynomial.

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- (2) is a stability result of Costello's conjecture over $\mathbb{C}$.

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Definition

f is a *d-multilinear form* if the variables are $(x_{i,j})_{i\in[d],j\in[n]}$ and every monomial has form $x_{1,i_1}x_{2,i_2}\ldots x_{d,i_d}$.

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Theorem (Costello, 2013)

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- Confirms the corrected conjecture for d-multilinear forms.

Global-local rank of the coefficient matrix

Setting: $f(\vec{x}) = \vec{x}^T A \vec{x}$, where $A \in \mathbb{R}^{n\times n}$ is symmetric.

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How to show f is close to rank-r or close to reducible?

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