Approximating Permanent of Random Matrices with Vanishing Mean: Made Better and Simpler

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Definition of permanent

- Given a matrix $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$, the permanent is given by

$$\mathrm{Per}(A) \triangleq \sum_{\sigma\in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

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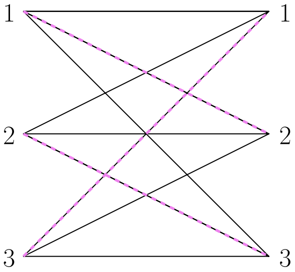

$$\begin{aligned}&\mathrm{Per}(A)\\ =\;& a_{1,1}a_{2,2}a_{3,3} + a_{1,1}a_{2,3}a_{3,2}\\ &+ a_{1,2}a_{2,1}a_{3,3} + \boldsymbol{a_{1,2}a_{2,3}a_{3,1}}\\ &+ a_{1,3}a_{2,1}a_{3,2} + a_{1,3}a_{2,2}a_{3,1}.\end{aligned}$$

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- There is a FPRAS to approximate (ϵ-approximation) permanent with non-negative entries. (Jerrum, Sinclair and Vigoda 01')
- Deciding the sign of the permanent is also #P-hard. (Aaronson 11')
- Seemingly impossible to derive a multiplicative approximation.

Average-case computational complexity

- Exact computing the permanent is #P-hard even for a constant portion of matrices with iid standard complex Gaussian entries. (Aaronson and Arkhipov 10')

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- Exact computing the permanent is #P-hard even for a constant portion of matrices with iid standard complex Gaussian entries. (Aaronson and Arkhipov 10')
- What about average-case approximations?
- Permanent-of-Gaussians Conjecture: approximating the permanent of Gaussian matrices is #P-hard. (Aaronson and Arkhipov 10')
- Easy for quantum computers but maybe hard for classical ones.

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- $\mu \to \infty$ is easy while $\mu = 0$ meets the Permanent-of-Gaussians Conjecture.
- To (dis-)prove the Permanent-of-Gaussians Conjecture, it is equivalent to consider $\mu = n^{-c}$ for some small constant $c > 0$. (Eldar and Mehraban 18')

Average-case approximation

- There is a quasi-polynomial scheme to approximate random permanents when $\mu = 1/\mathsf{poly}\log\log n$. (Eldar and Mehraban 18')

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- Can we improve the guarantee for μ?
- **Our result**: there is a quasi-polynomial scheme to approximate random permanents when $\mu = 1/\mathsf{poly}\log n$.

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- WLOG, consider matrix $M = J + zA$ where J the all-ones matrix and each entry of A is iid sampled from $\mathcal{D}$:

$$\mathbb{E}_{x\sim\mathcal{D}}[x] = 0,\ \mathrm{Var}_{x\sim\mathcal{D}}[x] = 1,\ \mathbb{E}_{x\sim\mathcal{D}}\left|x - \mu\right|^3 = \rho < \infty.$$

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- $z = 1/\mu = \mathsf{poly}\log n$.
- ρ is a constant when
  - μ-biased standard (complex) Gaussian: unit variance and mean μ;
  - biased Bernoulli: $-1+\mu$ w.p. $1/2$ while $1+\mu$ w.p. $1/2$.

Expansion of permanent

- $C_{n,k} \triangleq \{\{i_1, \cdots, i_k\} \subseteq [n]\}$;
- $P_{n,k} \triangleq \{(j_1, \cdots, j_k) \in [n]^k : j_\alpha \neq j_\beta \text{ for } \alpha \neq \beta\}$. $P_{n,n} = S_n$.

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$$\begin{aligned}
\frac{\mathrm{Per}(J + zA)}{n!} &= \frac{1}{n!} \sum_{\sigma \in P_{n,n}} \prod_{t=1}^{n} (1 + z a_{t,\sigma(t)}) \\
&= \frac{1}{n!} \sum_{\sigma \in P_{n,n}} \sum_{k=0}^{n} \sum_{\{i_1,\cdots,i_k\} \in C_{n,k}} \prod_{t=1}^{k} z a_{i_t,\sigma(i_t)} \\
&= \frac{1}{n!} \sum_{k=0}^{n} z^k \sum_{\{i_1,\cdots,i_k\} \in C_{n,k}} \sum_{\sigma \in P_{n,n}} \prod_{t=1}^{k} a_{i_t,\sigma(i_t)} \\
&= \frac{1}{n!} \sum_{k=0}^{n} z^k \sum_{\{i_1,\cdots,i_k\} \in C_{n,k}} \sum_{(j_1,\cdots,j_k) \in P_{n,k}} (n-k)! \prod_{t=1}^{k} a_{i_t,j_t}
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\frac{\mathrm{Per}(J + zA)}{n!} &= \frac{1}{n!} \sum_{k=0}^{n} z^k \sum_{\{i_1,\cdots,i_k\}\in C_{n,k}} \sum_{(j_1,\cdots,j_k)\in P_{n,k}} (n-k)! \prod_{t=1}^{k} a_{i_t,j_t} \\
&= \sum_{k=0}^{n} z^k \left(\frac{1}{n^{\underline{k}}} \sum_{\{i_1,\cdots,i_k\}\in C_{n,k}} \sum_{(j_1,\cdots,j_k)\in P_{n,k}} \prod_{t=1}^{k} a_{i_t,j_t} \right) \\
&= \sum_{k=0}^{n} \left(\frac{1}{n^{\underline{k}}} \sum_{\{j_1,\cdots,j_k\}\in C_{n,k}} \sum_{(i_1,\cdots,i_k)\in P_{n,k}} \prod_{t=1}^{k} a_{i_t,j_t} \right) z^k \\
&=: \sum_{k=0}^{n} a_k z^k.
\end{aligned}
$$

Basic idea

- Use $\sum_{k=0}^{t} a_k z^k$ to approximate $\frac{1}{n!}\operatorname{Per}(J + zA)$. Here $t = \ln \frac{n}{\epsilon}$ and the running time is simply $\mathcal{O}(n^{2t})$.

Basic idea

- Use $\sum_{k=0}^{t} a_k z^k$ to approximate $\frac{1}{n!}\operatorname{Per}(J + zA)$. Here $t = \ln\frac{n}{\epsilon}$ and the running time is simply $\mathcal{O}(n^{2t})$.
- It is sufficient to prove that w.p. $1 - o(1)$,

$$\left|\sum_{k=t+1}^{n} a_k z^k\right| < n^{-\gamma}\epsilon, \quad \left|\sum_{k=0}^{t} a_k z^k\right| > n^{-\gamma}.$$

A second moment bound

- $a_k = \frac{1}{n^{\underline{k}}} \sum_{\{j_1,\cdots,j_k\}\in C_{n,k}} \sum_{(i_1,\cdots,i_k)\in P_{n,k}} \prod_{t=1}^{k} a_{i_t,j_t}$.
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- $$\mathbb{E}[a_k \overline{a_l}] = \mathbb{1}_{k=l} \cdot \frac{1}{k!}.$$

A second moment bound

- $\mathbb{E}[a_k \overline{a_l}] = \mathbb{1}_{k=l} \cdot \frac{1}{k!}$, $z = \mathsf{poly}\log n$, $t = \log \frac{n}{\epsilon}$.

$$\mathrm{Var}\left[\sum_{k=t+1}^{n} a_k z^k\right] = \mathbb{E}\left[\sum_{k,l=t+1}^{n} a_k \overline{a_l} z^k \overline{z}^l\right] = \sum_{k=t+1}^{n} \frac{|z|^{2k}}{k!} = \left(\frac{\epsilon}{n}\right)^{\omega(1)}.$$

- Chebyshev's inequality.

Approximation of a_k's

- $C_{n,k} \triangleq \{\{i_1, \cdots, i_k\} \subseteq [n]\}$;
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$$\begin{aligned} a_k =& \frac{1}{n^k} \sum_{\{j_1,\cdots,j_k\} \in C_{n,k}} \sum_{\{i_1,\cdots,i_k\} \in P_{n,k}} \prod_{t=1}^{k} a_{i_t,j_t} \\ \approx& \frac{1}{n^k} \sum_{\{j_1,\cdots,j_k\} \in C_{n,k}} \sum_{i_1,\cdots,i_k \in [n]} \prod_{t=1}^{k} a_{i_t,j_t} \\ =& \frac{1}{n^k} \sum_{\{j_1,\cdots,j_k\} \in C_{n,k}} \left(\sum_{i=1}^{n} a_{i,j_1} \right) \cdots \left(\sum_{i=1}^{n} a_{i,j_k} \right) =: V_k. \end{aligned}$$

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- $\left| \sum_{k=0}^{t} a_k z^k - \sum_{k=0}^{t} V_k z^k \right| < n^{-0.1}$ with high probability.

Approximation of a_k's

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- C_j is "more concentrated" than $a_{i,j}$'s.
- Rewrite V_k with respect to C_j's:

$$\begin{aligned} V_k =& \frac{1}{n^k} \sum_{\{j_1,\cdots,j_k\} \in C_{n,k}} \left(\sum_{i=1}^n a_{i,j_1} \right) \left(\sum_{i=1}^n a_{i,j_2} \right) \cdots \left(\sum_{i=1}^n a_{i,j_k} \right) \\ =& \frac{1}{n^{k/2}} \sum_{\{j_1,\cdots,j_k\} \in C_{n,k}} C_{j_1} \cdots C_{j_k} \approx a_k. \end{aligned}$$

Properties of V_k's

- $n^{k/2} V_k = \sum_{\{j_1,\cdots,j_k\}\in C_{n,k}} C_{j_1}\cdots C_{j_k}$'s are the elementary symmetric polynomials.

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- Let $D_k \triangleq n^{-k/2} \sum_{j=1}^{n} C_j^k$.
- Newton's identities:

$$V_k = \frac{V_{k-1} V_1 - V_{k-2} D_2 + \sum_{i=2}^{k-1} (-1)^i V_{k-1-i} D_{i+1}}{k}.$$

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- $|D_k| \le n^{-0.1k}$ for all $k \ge 3$ with high probability.

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- $|D_k| \leq n^{-0.1k}$ for all $k \geq 3$ with high probability.
- A good approximation of V_k:

$$V'_k = \begin{cases} 1, & k = 0, \\ V_1, & k = 1, \\ \frac{V'_{k-1} V_1 - V'_{k-2} \xi}{k}, & k \geq 2. \end{cases}$$

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- $\left|\sum_{k=0}^{t} V_k z^k - \sum_{k=0}^{t} V'_k z^k\right| < n^{-0.08}$ with high probability.

Approximation of V_k's

- Represent V_k's with "probabilists' Hermite polynomials":

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- $$\sum_{k=0}^{t} V'_k z^k \approx \sum_{k=0}^{\infty} V'_k z^k = e^{V_1 z - \frac{\xi z^2}{2}} = (*).$$

  $|z| \leq (\ln n)^{0.05} \to (*)$ is small only if $|V_1|$ is relatively large, which is with probability $o(1)$.

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Summary

- $\frac{1}{n!}\operatorname{Per}(J + zA) = \sum_{k=0}^{n} a_k z^k$.
- $\left|\sum_{k=t+1}^{n} a_k z^k\right| < n^{-1}\epsilon$ with high probability.
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$$\sum_{k=0}^{t} a_k z^k \approx \sum_{k=0}^{t} V_k z^k \approx \sum_{k=0}^{t} V'_k z^k \approx \sum_{k=0}^{\infty} V'_k z^k = e^{V_1 z - \frac{\xi z^2}{2}}$$

is large ($> \frac{1}{2} n^{-0.01}$) with high probability.

Drawbacks and future questions

- From a_k to V_k, we need $\mathrm{Var}[z^k(a_k - V_k)]$ to be small, which does not hold for $\mu^{-1} = z = n^{0.0001}$.

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- Is there a (quasi-)polynomial algorithm for $\mu = 1/\mathsf{poly}(n)$?
- Is there a method to directly derive (approximately) the distribution of random permanents when $\mu = 1/\mathsf{poly}(n)$?
- Does the distribution of random permanents concentrated or rather anti-concentrated over $\mathbb{C}$?

Thank you.