Random Hasse diagrams & box-Delaunay graphs

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ETH Zürich

Joint work with Matthew Kwan and Lyuben Lichev

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Posets

Definition

A *poset* is a pair $(X, \preceq)$ where $\preceq$ is a partial order on X^2.

- Reflexivity: $a \preceq a$ for all $a \in X$;
- Antisymmetry: $a \preceq b$ and $b \preceq a$ imply $a = b$;
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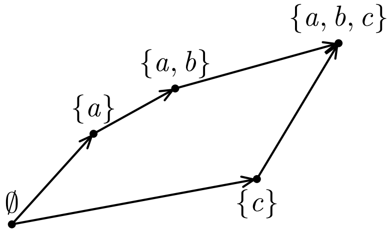
An example:



Comparability graphs

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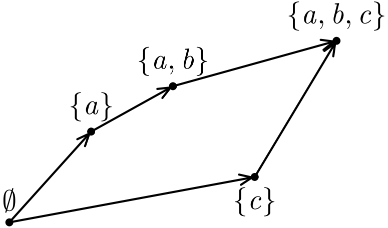

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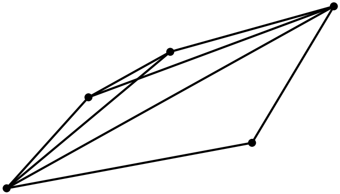

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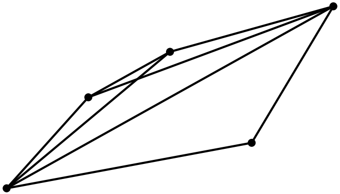

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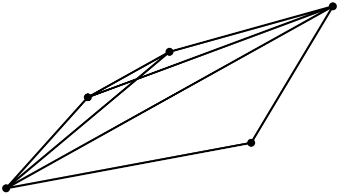

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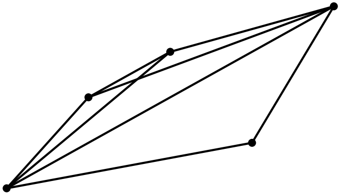

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  - clique number = chromatic number: by Mirsky's theorem.

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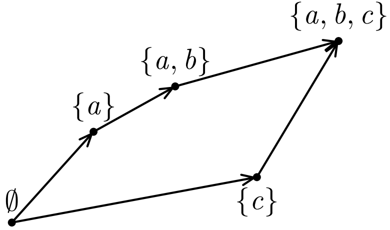

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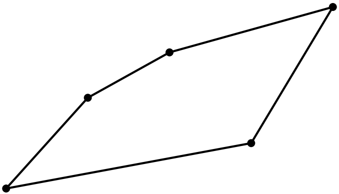

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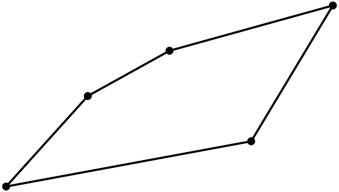

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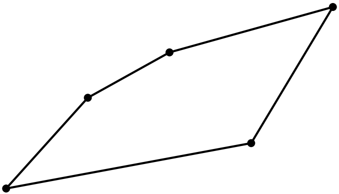

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What can we say about the clique number and the chromatic number of Hasse diagrams?

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- $\chi(G) \geq n/\alpha(G) = \omega(1)$.

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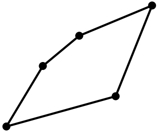

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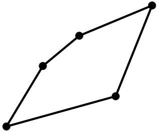

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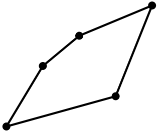

- Every finite poset can be embedded in $[0,1]^d$ for some d.
- Given a set P of n points in $[0,1]^d$, the d-dimensional Hasse diagram of P is the Hasse diagram corresponding to the (above) natural poset of P.

d-dimensional box-Delaunay graphs

Definition

Let P be a set of n points in $\mathbb{R}^d$. The *box-Delaunay graph* of P: connect $p, q \in P$ if the axis-parallel box enclosing p, q contains no other point in P.

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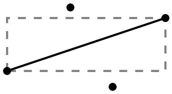

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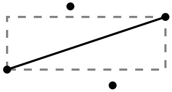

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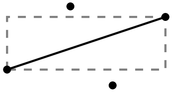

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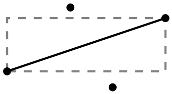

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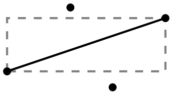

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- d-dim Hasse diagrams (box-Delaunay graphs) are geometric graphs that are defined **non-locally**!
  - (x, y) and (x', y') are more likely to be connected if $|x - x'| \cdot |y - y'|$ is small.

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- Both results work for chromatic numbers.

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  - $\alpha(G) = \Omega(n^{0.631})$ by Chan.

The End

Questions? Comments?