Exponential Erdős-Szekeres theorem for matrices

Zhihan Jin

ETH Zürich

Joint work with Recep Altar Çiçeksiz, Eero Räty and István Tomon

June 12, 2023

Erdős-Szekeres theorem

Theorem (Erdős-Szekeres '35)

Any sequence of $(n-1)^2+1$ *real numbers contains a monotone increasing or decreasing subsequence of length* n.

Theorem (Erdős-Szekeres '35)

Any sequence of $(n-1)^2+1$ *real numbers contains a monotone increasing or decreasing subsequence of length* n.

$$3 \quad 2 \quad 1 \quad 6 \quad 5 \quad 4 \quad 9 \quad 8 \quad 7 \quad 10$$

Erdős-Szekeres theorem

Theorem (Erdős-Szekeres '35)

Any sequence of $(n-1)^2+1$ real numbers contains a monotone increasing or decreasing subsequence of length n.

3 2 1 6 5 4 9 8 7 10

Erdős-Szekeres theorem

Theorem (Erdős-Szekeres '35)

Any sequence of $(n-1)^2+1$ real numbers contains a monotone increasing or decreasing subsequence of length n.

3 2 1 6 5 4 9 8 7 10

- $(n-1)^2+1$ is tight.

Erdős-Szekeres theorem

Theorem (Erdős-Szekeres '35)

Any sequence of $(n-1)^2+1$ real numbers contains a monotone increasing or decreasing subsequence of length n.

3 2 1 6 5 4 9 8 7 10

- $(n-1)^2+1$ is tight.
- Many beautiful proofs.

Erdős-Szekeres theorem

Theorem (Erdős-Szekeres '35)

Any sequence of $(n-1)^2+1$ real numbers contains a monotone increasing or decreasing subsequence of length n.

$$3 \quad 2 \quad \color{red}{1} \quad 6 \quad 5 \quad \color{red}{4} \quad 9 \quad 8 \quad \color{red}{7} \quad 10$$

- $(n-1)^2+1$ is tight.
- Many beautiful proofs.
- Different (natural) generalizations.

Multi-array Erdős-Szekeres theorem

Theorem (Burkill-Mirsky '73, Kalmanson '73)

Given r arrays each of length $n^{2^r} + 1$, there exist $n + 1$ indices that induce a monotone subsequence in each of the arrays.

Multi-array Erdős-Szekeres theorem

Theorem (Burkill-Mirsky '73, Kalmanson '73)

Given r arrays each of length $n^{2^r}+1$, there exist $n+1$ indices that induce a monotone subsequence in each of the arrays.

- Recursively apply Erdős-Szekeres theorem r times.

Multi-array Erdős-Szekeres theorem

Theorem (Burkill-Mirsky '73, Kalmanson '73)

Given r arrays each of length $n^{2^r}+1$, there exist $n+1$ indices that induce a monotone subsequence in each of the arrays.

- Recursively apply Erdős-Szekeres theorem r times.
- $n^{2^r}+1$ tight.

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

7	8	9
4	5	6
1	2	3

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

7	8	9
4	5	6
1	2	3

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

7	8	9
4	5	6
1	2	3

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

7	8	9
4	5	6
1	2	3

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

7	8	9
4	5	6
1	2	3

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

7	8	9
4	5	6
1	2	3

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

7	8	9
4	5	6
1	2	3

Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

7	8	9
4	5	6
1	2	3

Erdős-Szkeres-type question for matrices

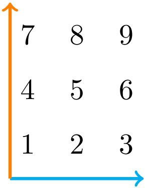
Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).



Erdős-Szkeres-type question for matrices

Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).

$$\begin{matrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 7 & 8 & 9 \\ 4 & 6 & 5 \\ 1 & 2 & 3 \end{matrix}$$

Erdős-Szkeres-type question for matrices

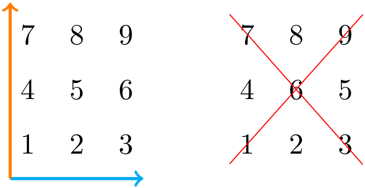
Question

How to generalize into the matrix setting?

Definition (Fishburn-Graham '93)

A matrix is *monotone* if

- all rows are monotone (in the same direction),
- all columns are monotone (in the same direction).



Erdős-Szkeres-type question for matrices

Question

Determine $M_2(n)$, the minimum N such that any $N \times N$ real matrix contains an $n \times n$ monotone submatrix.

Erdős-Szkeres-type question for matrices

Question

Determine $M_2(n)$, the minimum N such that any $N \times N$ real matrix contains an $n \times n$ monotone submatrix.

Theorem (Fishburn-Graham '93)

$n^{n/2} \leq M_2(n) \leq \mathrm{twr}_5(O(n))$.

Erdős-Szkeres-type question for matrices

Question

Determine $M_2(n)$, the minimum N such that any $N \times N$ real matrix contains an $n \times n$ monotone submatrix.

Theorem (Fishburn-Graham '93)

$n^{n/2} \leq M_2(n) \leq \mathrm{twr}_5(O(n))$.

Theorem (Bucić-Sudakov-Tran '22)

$M_2(n) \leq 2^{2^{(2+o(1))n}}$.

Erdős-Szkeres-type question for matrices

Question

Determine $M_2(n)$, the minimum N such that any $N \times N$ real matrix contains an $n \times n$ monotone submatrix.

Theorem (Fishburn-Graham '93)

$n^{n/2} \leq M_2(n) \leq \mathrm{twr}_5(O(n))$.

Theorem (Bucić-Sudakov-Tran '22)

$M_2(n) \leq 2^{2^{(2+o(1))n}}$.

- The lower bound is due to a probabilistic argument.

Lexicographic matrices

- Matrix A is *lex-increasing* if

$$A_{i,j} \leq A_{k,\ell} \Leftrightarrow i < k \text{ or } i = k \text{ and } j < \ell.$$

Lexicographic matrices

- Matrix A is *lex-increasing* if

$$A_{i,j} \leq A_{k,\ell} \Leftrightarrow i < k \text{ or } i = k \text{ and } j < \ell.$$

Definition

A matrix is *lex-monotone* if one can rotate and/or mirror it to get a lex-increasing matrix.

Lexicographic matrices

- Matrix A is *lex-increasing* if

$$A_{i,j} \leq A_{k,\ell} \Leftrightarrow i < k \text{ or } i = k \text{ and } j < \ell.$$

Definition

A matrix is *lex-monotone* if one can rotate and/or mirror it to get a lex-increasing matrix.

7	8	9
4	5	6
1	2	3

Lexicographic matrices

- Matrix A is *lex-increasing* if

$$A_{i,j} \leq A_{k,\ell} \Leftrightarrow i < k \text{ or } i = k \text{ and } j < \ell.$$

Definition

A matrix is *lex-monotone* if one can rotate and/or mirror it to get a lex-increasing matrix.

7	8	9		3	6	9
4	5	6		2	5	8
1	2	3		1	4	7

Lexicographic matrices

- Matrix A is *lex-increasing* if

$$A_{i,j} \leq A_{k,\ell} \Leftrightarrow i < k \text{ or } i = k \text{ and } j < \ell.$$

Definition

A matrix is *lex-monotone* if one can rotate and/or mirror it to get a lex-increasing matrix.

$$\begin{matrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{matrix} \qquad \begin{matrix} 6 & 8 & 9 \\ 2 & 5 & 7 \\ 1 & 3 & 4 \end{matrix}$$

Lexicographic matrices

- Matrix A is *lex-increasing* if

$$A_{i,j} \leq A_{k,\ell} \Leftrightarrow i < k \text{ or } i = k \text{ and } j < \ell.$$

Definition

A matrix is *lex-monotone* if one can rotate and/or mirror it to get a lex-increasing matrix.

$$\begin{matrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{matrix} \qquad \begin{matrix} 6 & 8 & 9 \\ 2 & 5 & 7 \\ 1 & 3 & 4 \end{matrix}$$

Lexicographic matrices

- Matrix A is *lex-increasing* if

$$A_{i,j} \le A_{k,\ell} \Leftrightarrow i < k \text{ or } i = k \text{ and } j < \ell.$$

Definition

A matrix is *lex-monotone* if one can rotate and/or mirror it to get a lex-increasing matrix.

$$\begin{matrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{matrix} \qquad \begin{matrix} 6 & 8 & 9 \\ 2 & 5 & 7 \\ 1 & 3 & 4 \end{matrix}$$

- There are 8 different "types" of lex-monotone matrices.

Lexicographic matrices

Question

Determine $L_2(n)$, the minimum N such that any $N \times N$ real matrix contains a $n \times n$ lex-monotone submatrix.

Lexicographic matrices

Question

Determine $L_2(n)$, the minimum N such that any $N \times N$ real matrix contains a $n \times n$ lex-monotone submatrix.

Theorem (Fishburn-Graham '93)

$L_2(n) \leq M_2(2n^2 - 5n + 4) = \mathrm{twr}_5(O(n^2))$.

Lexicographic matrices

Question

Determine $L_2(n)$, the minimum N such that any $N \times N$ real matrix contains a $n \times n$ lex-monotone submatrix.

Theorem (Fishburn-Graham '93)

$L_2(n) \leq M_2(2n^2 - 5n + 4) = \mathrm{twr}_5(O(n^2))$.

Theorem (Bucić-Sudakov-Tran '22)

$L_2(n) \leq M_2(2n^2 - 5n + 4) = 2^{2^{O(n^2)}}$.

Our results

Theorem (Çiçeksiz, J., Räty, Tomon '23)

An $O(n^4) \times 2^{O(n^4(\log n)^2)}$ *real matrix contains a* $n \times n$ *monotone submatrix. In particular,* $M_2(n) = 2^{O(n^4(\log n)^2)}$.

Our results

Theorem (Çiçeksiz, J., Räty, Tomon '23)

An $O(n^4) \times 2^{O(n^4(\log n)^2)}$ *real matrix contains a* $n \times n$ *monotone submatrix. In particular,* $M_2(n) = 2^{O(n^4(\log n)^2)}$.

Theorem (Çiçeksiz, J., Räty, Tomon '23)

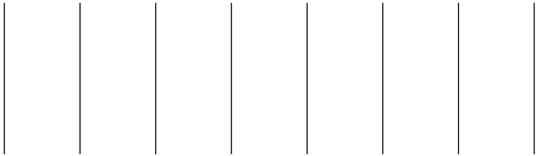
$L_2(n) \leq M_2(2n^2 - 5n + 4) \leq 2^{O(n^8(\log n)^2)}$.

A pigeonhole argument

Goal: show that $M_2(n) = 2^{2^{O(n)}}$.

A pigeonhole argument

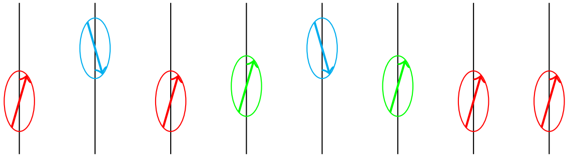
Goal: show that $M_2(n) = 2^{2^{O(n)}}$.



- Restricted to a $n^2 \times N$ matrix with $N \approx 2^{2^{O(n)}}$.

A pigeonhole argument

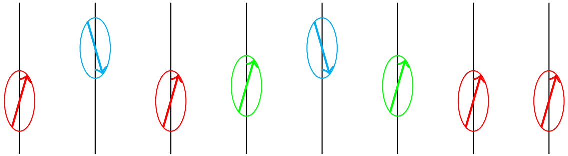
Goal: show that $M_2(n) = 2^{2^{O(n)}}$.



- Restricted to a $n^2 \times N$ matrix with $N \approx 2^{2^{O(n)}}$.
- Apply Erdős-Szekeres theorem on each column to get a monotone sub-column of size n.

A pigeonhole argument

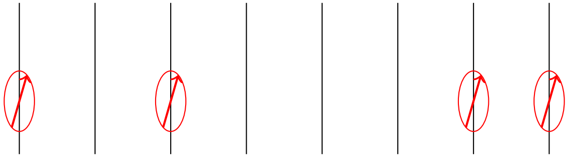
Goal: show that $M_2(n) = 2^{2^{O(n)}}$.



- Restricted to a $n^2 \times N$ matrix with $N \approx 2^{2^{O(n)}}$.
- Apply Erdős-Szekeres theorem on each column to get a monotone sub-column of size n.
- Apply pigeonhole principle on the index sets of the monotone sub-columns and the direction.

A pigeonhole argument

Goal: show that $M_2(n) = 2^{2^{O(n)}}$.



- Restricted to a $n^2 \times N$ matrix with $N \approx 2^{2^{O(n)}}$.
- Apply Erdős-Szekeres theorem on each column to get a monotone sub-column of size n.
- Apply pigeonhole principle on the index sets of the monotone sub-columns and the direction.
- Get a column-monotone matrix of size $n \times N/2\binom{n^2}{n} \approx n \times 2^{2^{O(n)}}$.

A pigeonhole argument

Goal: show that $M_2(n) = 2^{2^{O(n)}}$.

A pigeonhole argument

Goal: show that $M_2(n) = 2^{2^{O(n)}}$.

- Our matrix is $n \times 2^{2^{O(n)}}$ and column-monotone.

A pigeonhole argument

Goal: show that $M_2(n) = 2^{2^{O(n)}}$.

- Our matrix is $n \times 2^{2^{O(n)}}$ and column-monotone.
- Apply multi-array Erdős-Szekeres.

A pigeonhole argument

Goal: show that $M_2(n) = 2^{2^{O(n)}}$.

- Our matrix is $n \times 2^{2^{O(n)}}$ and column-monotone.
- Apply multi-array Erdős-Szekeres.
  - Row-monotone submatrix of size $n \times n$.

A pigeonhole argument

Goal: show that $M_2(n) = 2^{2^{O(n)}}$.

- Our matrix is $n \times 2^{2^{O(n)}}$ and column-monotone.
- Apply multi-array Erdős-Szekeres.
  - Row-monotone submatrix of size $n \times n$.
  - This requires n^{2^n} columns.

A pigeonhole argument

Goal: show that $M_2(n) = 2^{2^{O(n)}}$.

- Our matrix is $n \times 2^{2^{O(n)}}$ and column-monotone.
- Apply multi-array Erdős-Szekeres.
  - Row-monotone submatrix of size $n \times n$.
  - This requires n^{2^n} columns.
- A $n \times n$ monotone submatrix!

Find row-monotone matrices

$\mathbb{R}^{n^2 \times N} \ni A$

Find row-monotone matrices

$$\mathbb{R}^{n^2 \times N} \ni A \quad \xrightarrow[\text{column-monotone}]{\text{pay } \exp(n) \text{ in columns}} \quad B$$

Find row-monotone matrices

$$\mathbb{R}^{n^2\times N} \ni A \quad \xrightarrow[\text{column-monotone}]{\text{pay } \exp(n) \text{ in columns}} \quad B \quad \xrightarrow[\text{row-monotone}]{\text{pay } 2^{2^n} \text{ in columns}} \quad C$$

Find row-monotone matrices

$$\mathbb{R}^{n^2 \times N} \ni A \quad \xrightarrow[\text{column-monotone}]{\text{pay } \exp(n) \text{ in columns}} \quad B \quad \xrightarrow[\text{row-monotone}]{\text{pay } 2^{2^n} \text{ in columns}} \quad C$$

Question

Can we do better if we start with more rows (instead of n^2)?

Find row-monotone matrices

$$\mathbb{R}^{n^2 \times N} \ni A \quad \xrightarrow[\text{column-monotone}]{\text{pay } \exp(n) \text{ in columns}} \quad B \quad \xrightarrow[\text{row-monotone}]{\text{pay } 2^{2^n} \text{ in columns}} \quad C$$

Question

Can we do better if we start with more rows (instead of n^2)?

More specifically,

Question

Is there a $n \times n$ row-monotone submatrix in a $2^{o(n)} \times 2^{2^{o(n)}}$ matrix?

Find row-monotone matrices

Theorem (Çiçeksiz, J., Räty, Tomon '23)

Any $10n^2 \times 2^{\tilde{O}(n^4)}$ *matrix contains an* $n \times n$ *row-monotone submatrix.*

Find row-monotone matrices

Theorem (Çiçeksiz, J., Räty, Tomon '23)

Any $10n^2 \times 2^{\tilde{O}(n^4)}$ *matrix contains an* $n \times n$ *row-monotone submatrix.*

Theorem (Çiçeksiz, J., Räty, Tomon '23)

There exists an $\Omega(n^2) \times 2^{2^{\tilde{\Omega}(n)}}$ *matrix with no* $n \times n$ *row-monotone submatrix.*

Find row-monotone matrices

Theorem (Çiçeksiz, J., Räty, Tomon '23)

Any $10n^2 \times 2^{\tilde{O}(n^4)}$ matrix contains an $n \times n$ row-monotone submatrix.

Theorem (Çiçeksiz, J., Räty, Tomon '23)

There exists an $\Omega(n^2) \times 2^{2^{\tilde{\Omega}(n)}}$ matrix with no $n \times n$ row-monotone submatrix.

- A sharp transition!

A single-exponential improvement

Theorem (Çiçeksiz, J., Räty, Tomon '23)

Any $10n^2 \times 2^{\tilde{O}(n^4)}$ *matrix contains an* $n \times n$ *row-monotone submatrix.*

A single-exponential improvement

Theorem (Çiçeksiz, J., Räty, Tomon '23)

Any $10n^2 \times 2^{\tilde{O}(n^4)}$ *matrix contains an* $n \times n$ *row-monotone submatrix.*

Corollary

Any $O(n^4) \times 2^{\tilde{O}(n^4)}$ *matrix contains an* $n \times n$ *monotone submatrix.*

A single-exponential improvement

Theorem (Çiçeksiz, J., Räty, Tomon '23)

Any $10n^2 \times 2^{\tilde{O}(n^4)}$ *matrix contains an* $n \times n$ *row-monotone submatrix.*

Corollary

Any $O(n^4) \times 2^{\tilde{O}(n^4)}$ *matrix contains an* $n \times n$ *monotone submatrix.*

- $M_2(n) \leq 2^{\tilde{O}(n^4)}$.

Future directions

- We know that $n^{n/2} \leq M_2(n) \leq 2^{\tilde{O}(n^4)}$.

Future directions

- We know that $n^{n/2} \leq M_2(n) \leq 2^{\tilde{O}(n^4)}$.

Question

What is the exponent of $M_2(n)$?

Future directions

- An $n \times m^{2^n}$ matrix $\Rightarrow$ an $n \times m$ row-monotone submatrix.

Future directions

- An $n \times m^{2^n}$ matrix $\Rightarrow$ an $n \times m$ row-monotone submatrix.
- An $O(n^2) \times 2^{\tilde{O}(n^4)}$ matrix $\Rightarrow$ an $n \times n$ row-monotone submatrix.

Future directions

- An $n \times m^{2^n}$ matrix $\Rightarrow$ an $n \times m$ row-monotone submatrix.
- An $O(n^2) \times 2^{\tilde{O}(n^4)}$ matrix $\Rightarrow$ an $n \times n$ row-monotone submatrix.
- An $m^{2m}n \times m^2$ matrix $\Rightarrow$ an $n \times m$ row-monotone submatrix.

Future directions

- An $n \times m^{2^n}$ matrix $\Rightarrow$ an $n \times m$ row-monotone submatrix.
- An $O(n^2) \times 2^{\tilde{O}(n^4)}$ matrix $\Rightarrow$ an $n \times n$ row-monotone submatrix.
- An $m^{2m}n \times m^2$ matrix $\Rightarrow$ an $n \times m$ row-monotone submatrix.

Question

Are there other regimes?

Future directions

- $K_n \square K_n$ is a graph with vertex set $[n]^2$ where $(x, y) \sim (x', y')$ if $x = x'$ or $y = y'$.

Future directions

- $K_n \square K_n$ is a graph with vertex set $[n]^2$ where $(x, y) \sim (x', y')$ if $x = x'$ or $y = y'$.
- A coloring $c : E(K_n \square K_n) \to [r]$ is *monochromatic* if all edges of form $((x, y), (x, y'))$ share some color c_1 and all edges of form $((x, y), (x', y))$ share some color c_2.

Future directions

- $K_n \square K_n$ is a graph with vertex set $[n]^2$ where $(x, y) \sim (x', y')$ if $x = x'$ or $y = y'$.
- A coloring $c : E(K_n \square K_n) \to [r]$ is *monochromatic* if all edges of form $((x, y), (x, y'))$ share some color c_1 and all edges of form $((x, y), (x', y))$ share some color c_2.
  - It is possible that $c_1 \neq c_2$.

Future directions

- $K_n \Box K_n$ is a graph with vertex set $[n]^2$ where $(x, y) \sim (x', y')$ if $x = x'$ or $y = y'$.
- A coloring $c : E(K_n \Box K_n) \to [r]$ is *monochromatic* if all edges of form $((x, y), (x, y'))$ share some color c_1 and all edges of form $((x, y), (x', y))$ share some color c_2.
  - It is possible that $c_1 \neq c_2$.
- Given an $n \times n$ matrix A, color $((x, y), (x, y'))$ red if $A_{x,y} < A_{x,y'}$ and blue otherwise; color $((x, y), (x', y))$ red if $A_{x,y} < A_{x',y}$ and blue otherwise.

Future directions

- $K_n \square K_n$ is a graph with vertex set $[n]^2$ where $(x, y) \sim (x', y')$ if $x = x'$ or $y = y'$.
- A coloring $c : E(K_n \square K_n) \to [r]$ is *monochromatic* if all edges of form $((x, y), (x, y'))$ share some color c_1 and all edges of form $((x, y), (x', y))$ share some color c_2.
  - It is possible that $c_1 \neq c_2$.
- Given an $n \times n$ matrix A, color $((x, y), (x, y'))$ red if $A_{x,y} < A_{x,y'}$ and blue otherwise; color $((x, y), (x', y))$ red if $A_{x,y} < A_{x',y}$ and blue otherwise.
  - A is monotone $\Leftrightarrow$ the coloring is monochromatic.

Future directions

Question

Determine $f(n; r)$, the minimum N such that any r-coloring of $E(K_N \square K_N)$ contains a monochromatic $K_n \square K_n$.

Future directions

Question

Determine $f(n;r)$, the minimum N such that any r-coloring of $E(K_N \square K_N)$ contains a monochromatic $K_n \square K_n$.

- $f(n;r) \leq r^{r^{O(rn^2)}}$ by Girão, Kronenberg and Scott.

Future directions

Question

Determine $f(n;r)$, the minimum N such that any r-coloring of $E(K_N \square K_N)$ contains a monochromatic $K_n \square K_n$.

- $f(n;r) \leq r^{r^{O(rn^2)}}$ by Girão, Kronenberg and Scott.
  - A similar pigeonhole argument works.

Future directions

Question

Determine $f(n;r)$, the minimum N such that any r-coloring of $E(K_N \square K_N)$ contains a monochromatic $K_n \square K_n$.

- $f(n;r) \leq r^{r^{O(rn^2)}}$ by Girão, Kronenberg and Scott.
  - A similar pigeonhole argument works.
- Our method does not seem to generalize.

Future directions

- Similar questions in higher dimensions?

Future directions

- Similar questions in higher dimensions?

Question

Determine $M_d(n)$, the minimum N such that any d-dimensional $N \times \cdots \times N$ real matrix contains a $n \times \cdots \times n$ monotone submatrix.

- $n^{\Omega(n^2)} \leq M_3(n) \leq 2^{2^{n^2}}$.

Future directions

- Similar questions in higher dimensions?

Question

Determine $M_d(n)$, the minimum N such that any d-dimensional $N \times \cdots \times N$ real matrix contains a $n \times \cdots \times n$ monotone submatrix.

- $n^{\Omega(n^2)} \leq M_3(n) \leq 2^{2^{n^2}}$.
- Our method does not work.

Future directions

- Similar questions in higher dimensions?

Question

Determine $M_d(n)$, the minimum N such that any d-dimensional $N \times \cdots \times N$ real matrix contains a $n \times \cdots \times n$ monotone submatrix.

- $n^{\Omega(n^2)} \leq M_3(n) \leq 2^{2^{n^2}}$.
- Our method does not work.
  - Our strategy: first have $\{x, y\}$-monotone, and then z-monotone.

Future directions

- Similar questions in higher dimensions?

Question

Determine $M_d(n)$, the minimum N such that any d-dimensional $N \times \cdots \times N$ real matrix contains a $n \times \cdots \times n$ monotone submatrix.

- $n^{\Omega(n^2)} \leq M_3(n) \leq 2^{2^{n^2}}$.
- Our method does not work.
  - Our strategy: first have $\{x, y\}$-monotone, and then z-monotone.
  - $\exists\ 2^{n/2} \times 2^{n/2} \times 2^{2^n}$ matrix with no $n \times n \times 2$ z-monotone submatrix.

Future directions

- Similar questions in higher dimensions?

Question

Determine $M_d(n)$, the minimum N such that any d-dimensional $N \times \cdots \times N$ real matrix contains a $n \times \cdots \times n$ monotone submatrix.

- $n^{\Omega(n^2)} \leq M_3(n) \leq 2^{2^{n^2}}$.
- Our method does not work.
  - Our strategy: first have $\{x, y\}$-monotone, and then z-monotone.
  - $\exists\ 2^{n/2} \times 2^{n/2} \times 2^{2^n}$ matrix with no $n \times n \times 2$ z-monotone submatrix.
  - One needs to pay 2^{2^n}.

Future directions

Question

Determine $L_d(n)$, the minimum N such that any d-dimensional $N \times \cdots \times N$ real matrix contains a $n \times \cdots \times n$ lex-monotone submatrix.

Future directions

Question

Determine $L_d(n)$, the minimum N such that any d-dimensional $N \times \cdots \times N$ real matrix contains a $n \times \cdots \times n$ lex-monotone submatrix.

- $L_d(n) \leq M_d(2^{n^{d-2}}) = 2^{2^{2^{n^{d-1}}}}$ by Bucić-Sudakov-Tran and Girão-Kronenberg-Scott.

Future directions

Question

Determine $L_d(n)$, the minimum N such that any d-dimensional $N \times \cdots \times N$ real matrix contains a $n \times \cdots \times n$ lex-monotone submatrix.

- $L_d(n) \leq M_d(2^{n^{d-2}}) = 2^{2^{2^{n^{d-1}}}}$ by Bucić-Sudakov-Tran and Girão-Kronenberg-Scott.

The End

Questions? Comments?