Random Hasse diagrams & box-Delaunay graphs

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Joint work with Matthew Kwan and Lyuben Lichev

April 27, 2025

Posets

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A *poset* is a pair $(X, \preceq)$ where $\preceq$ is a partial order on X^2.

- Reflexivity: $a \preceq a$ for all $a \in X$;
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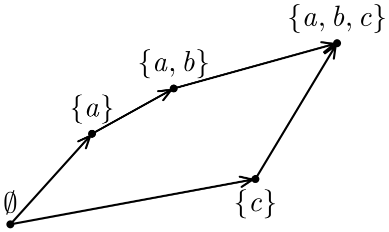
An example:



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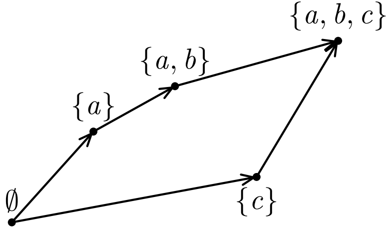

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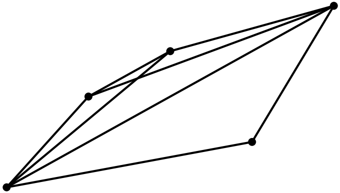

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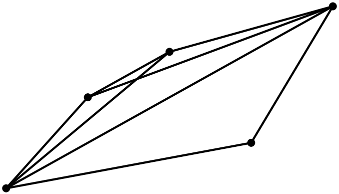

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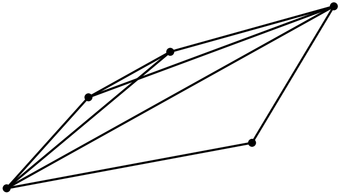

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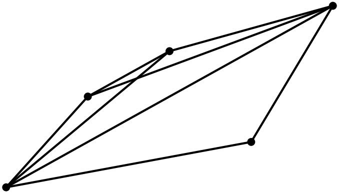

- Form an important family of perfect graphs.
  - Hereditary: deleting a vertex remains a comparability graph.
  - clique number = chromatic number: by Mirsky's theorem.

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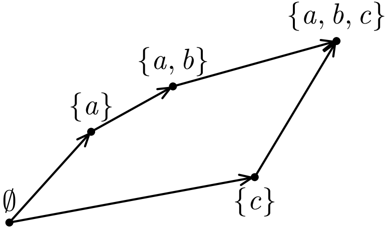

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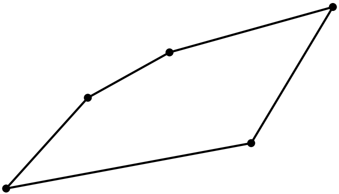

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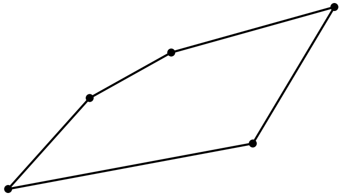

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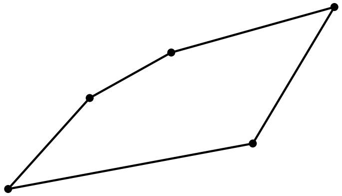

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Question

What can we say about the clique number and the chromatic number of Hasse diagrams?

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- $\chi(G) \geq n/\alpha(G) = \omega(1)$.

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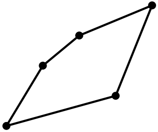

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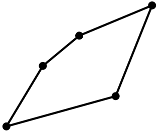

- Every finite poset can be embedded in $[0,1]^d$ for some d.

General Delaunay graphs

Definition

Let $\mathcal{C}$ be a family of convex bodies in $\mathbb{R}^d$. Given $P \subset [0,1]^d$, the *Delaunay graph* of P (w.r.t. $\mathcal{C}$): connect $p, q \in P$ if some $C \in \mathcal{C}$ contains only p, q in P.

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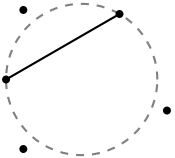

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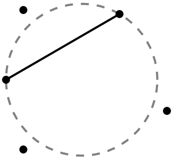

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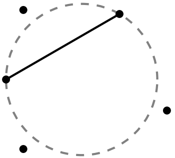

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- Planar $\Rightarrow \chi(G) \leq 4$.

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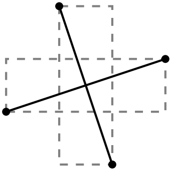

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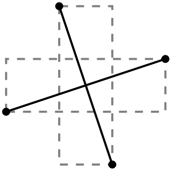

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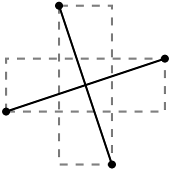

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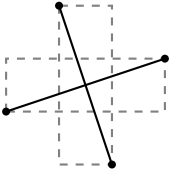

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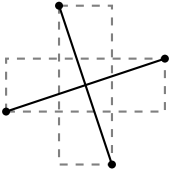

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  - (x, y) and (x', y') are more likely to be connected if $|x - x'| \cdot |y - y'|$ is small.

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- $\Delta(G) \approx (\log n)^{d-1}$ whp.
- Expect $\alpha(G) \approx n \log\log n / (\log n)^{d-1}$ whp.

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Suppose $P \subseteq [0,1]^2$ *is* uniformly at random. *Then, whp, the Hasse diagram (or the box-Delaunay graph) of* P *has independence number* $O(n(\log\log n)^2/\log n)$.

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- He considered random points with modification.
- The method does not work for Hasse diagrams.
- Worse than what we expected: $n\log\log n/(\log n)^{d-1}$.

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- Both results work for chromatic numbers.

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Further directions

- We showed the independence number satisfies $\alpha(G) = \Omega(n \log\log n/(\log n)^{d-1})$ and $\alpha(G) = O(n(\log\log n)^{2d-2}/(\log n)^{d-1})$.
- Which is closer to the truth?
- What about the deterministic case for dimension d?

Question

Fix $d \geq 2$.
Is it true that any Hasse diagram (or box-Delaunay graph) of a d-dimensional point set has independence number $n^{1-o(1)}$?

- Is it true when $d = 2$?
  - $\alpha(G) = \Omega(n^{0.618})$ by Ajwani–Elbassioni–Govindarajan–Ray.

The End

Questions? Comments?