From Small Eigenvalues to Large Cuts and Chowla's Cosine Problem

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ISTA

Joint work with Aleksa Milojević, István Tomon and Shengtong Zhang

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  - Many graph parameters are NP-hard to compute/approximate: the chromatic number, the clique number, etc.

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- We will focus on λ_n.

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- $G(n, 1/2)$: $\omega(G) \approx 2 \log n$ (small); $|\lambda_n| \approx 2\sqrt{n}$ (not too small).

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(1) *G has a clique of size $\approx d/|\lambda_n|^4$ whenever $|\lambda_n| < d^{0.1-o(1)}$.*

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- In (2), $1/4$ on the exponent is the best possible.

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- A polynomial stability for Erwards' theorem.
- The first polynomial improvement for the Alon–Bollobás–Krivelevich–Sudakov conjecture.

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$$\begin{aligned} \mathrm{surp}(G) &= \max_{x \in \{\pm 1\}^n} -\frac{1}{4} x^T A_G x \\ &\leq \max_{x \in \{\pm 1\}^n} -\frac{1}{4} \lambda_n \cdot \|x\|_2^2 \end{aligned}$$

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$$\begin{aligned} \mathrm{surp}(G) &= \max_{x \in \{\pm 1\}^n} -\frac{1}{4} x^T A_G x \\ &\leq \max_{x \in \{\pm 1\}^n} -\frac{1}{4} \lambda_n \cdot \|x\|_2^2 = |\lambda_n| n/4. \end{aligned}$$

Surplus and eigenvalues

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How to connect $\mathrm{surp}(G)$ and λ_n?

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- Extra work...

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Want to know more?

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- Consider a Cayley graph G:
  - $V(G) = \mathbb{Z}/p\mathbb{Z}$; $x \sim y$ if $x - y \in A$ or $y - x \in A$.

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  - Using the symmetry of G, we can show this is impossible.