Difference-isomorphic graph families

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Joint work with Lior Gishboliner and Benny Sudakov

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If $\mathcal{A}$ is a family of distinct subsets of $[n]$ s.t. no set is contained in the other, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

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- If $|\mathcal{A}| \approx \binom{n-1}{r-1}$, then $\mathcal{A}$ is "close" to the extremal example.

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Conjecture (Simonovits-Sós 1976)

If $\mathcal{G}$ is a family of graphs on $[n]$ s.t. any two graphs in $\mathcal{G}$ share a common triangle ($\triangle$), then $|\mathcal{G}| \leq 2^{\binom{n}{2}-3}$.

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- $\triangle \Rightarrow C_n$, $|G| \leq 2^{\binom{n}{2}-n}$ by Leader, Ranđelović and Tan.

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Conjecture (Gowers, 2009)

$\forall \delta > 0 \; \forall n \gg_\delta 1$, if $\mathcal{A}$ is a family of $n \times n$ matrices with entries in $\{1, \ldots, k\}$ with $|\mathcal{A}| > \delta \cdot k^{n^2}$, then $\exists A_1, \ldots, A_k \in \mathcal{A}, X \subseteq [n]$ s.t. $A_{i+1} - A_i = \mathbb{1}_{X \times X}$ for all i.

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Question(Alon 2023+)

When $\mathcal{H}$ contains all the K_4's, does $|\mathcal{G}| = o(2^{\binom{n}{2}})$?

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  - If so, maybe $|\mathcal{G}| \leq n^{O(n)}$ as there are $n!$ isomorphisms.

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  - $(G_1 \setminus G_2)\big|_{e,f} = (G_2 \setminus G_1)\big|_{e,f} = \emptyset$ or
    $(G_1 \setminus G_2)\big|_{e,f} = \{e\}, (G_2 \setminus G_1)\big|_{e,f} = \{f\}$ or
    $(G_1 \setminus G_2)\big|_{e,f} = \{f\}, (G_2 \setminus G_1)\big|_{e,f} = \{e\}$.
  - $\psi\Big((G_1 \setminus G_2)\big|_{e,f}\Big) = (G_2 \setminus G_1)\big|_{e,f}$.
- $\psi(G_1 \setminus G_2) = G_2 \setminus G_1$.
- $|\mathcal{G}| = 2^{\#(\text{pairs of edges})} = 2^{2\binom{k}{2}} = 2^{\frac{1}{2}(\binom{n}{2} - \frac{n}{2})} \gg n^{O(n)}$.
  - $\frac{n}{2}$ comes from edges $u_1 v_1, u_2 v_2, \dots, u_k v_k$.

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- All the above can be extended to families of r-graphs ($n \gg r$).

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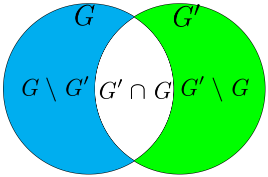

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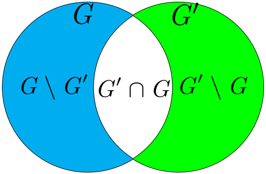

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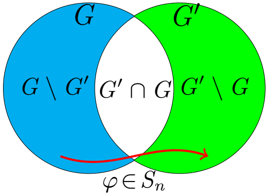

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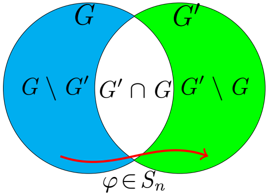

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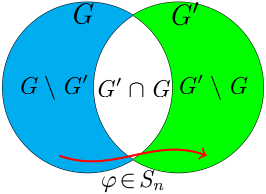

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  - Many $G_1 \overset{\varphi}{\cong} G_2$ in $\mathcal{G} \Rightarrow \exists$ many graphs in $\mathcal{G}$ forming a φ-clique.

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(β) *or* $\mathcal{G}$ *contains a "large"* ψ*-clique for some involution* ψ*.*

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A sketch of the upper bound proof

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(1) $e_\psi(N_\varphi(G)) < M^2 \cdot n^{-10n}$ *unless* φ, ψ *are "close".*

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- Assume (α) happens ((β) is more complicated).

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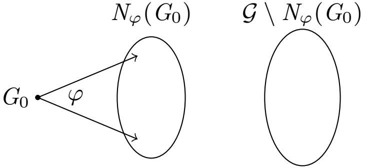

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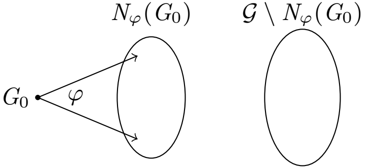

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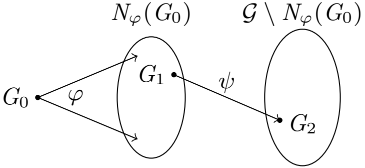

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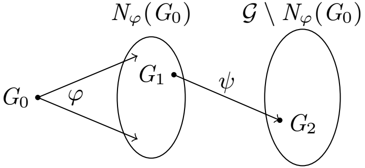

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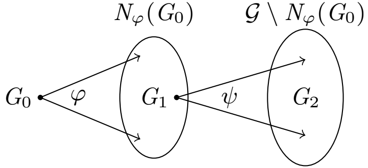

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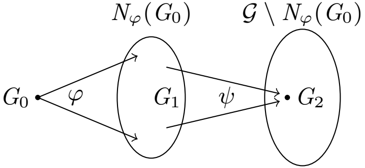

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- For ψ "close" to φ, $\#\{G_1 \overset{\psi}{\cong} G_2\} \leq |N_\varphi(G_0)| \cdot M \cdot 2^{-n/200}$.
- For other ψ, $\#\{G_1 \overset{\psi}{\cong} G_2\} \leq |\mathcal{G} \setminus N_\varphi(G_0)| \cdot M \cdot n^{-10n}$.

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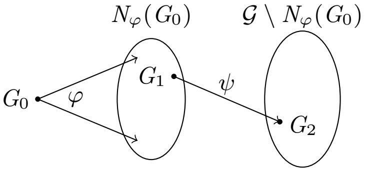
Condition: $M = 2^{\frac{1}{2}(\binom{n}{2}-\frac{n}{2})}, |\mathcal{G}|/n! \leq |N_\varphi(G_0)| < M \cdot 2^{-n/100}$.



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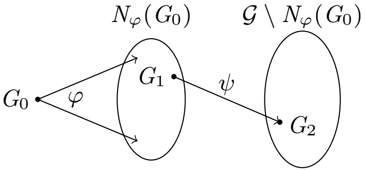

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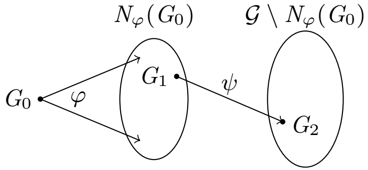

$$
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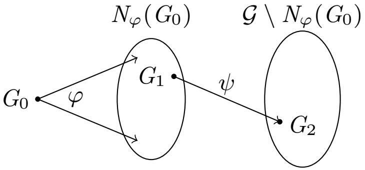

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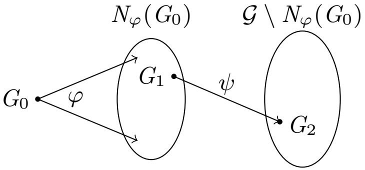

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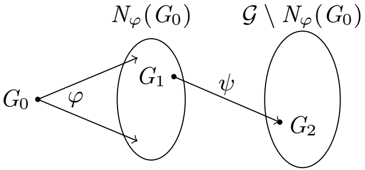

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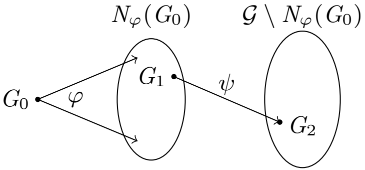

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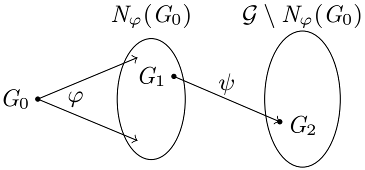

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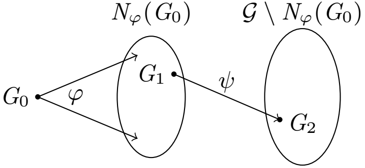

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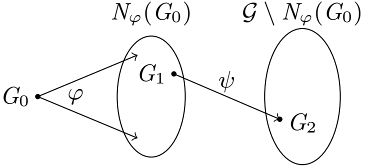

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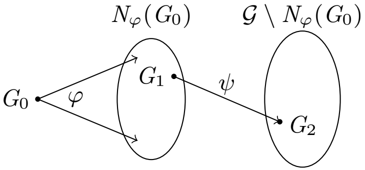

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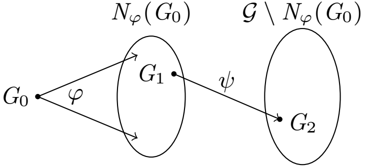

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$$\Longrightarrow |\mathcal{G}| \leq M \cdot 2^{-n/400}!$$

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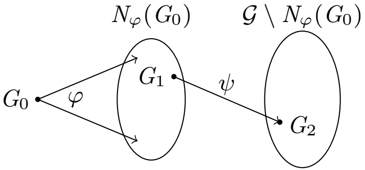

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\Longrightarrow\ & |\mathcal{G}| \le M \cdot 2^{-n/400}!
\end{aligned}$$

Nice!

Further directions

Further directions

Any suggestions?

The End

Questions? Comments?