Minimum Degree Removal Lemma Thresholds

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Joint work with Lior Gishboliner and Benny Sudakov

The chromatic threshold

Conjecture (Erdős-Simonovits '73)

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Theorem (Brandt-Thomasse '11)

If G is K_3-free and $\delta(G) > \frac{n}{3}$ then $\chi(G) \leq 4$.

- This is tight by Hajnal's construction.

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The Allen-Böttcher-Griffiths-Kohayakawa-Morris Theorem '13

Determines $\delta_\chi(H)$ for every H. If $\chi(H) = r$ then

$$\delta_\chi(H) \in \left\{ \frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1} \right\}.$$

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$\delta_{hom}(\{C_3, C_5 \ldots, C_{2k+1}\}) = \frac{1}{2k+1}$ *and* $\delta_{hom}(C_{2k+1}) \leq \frac{1}{2k+1}$.

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Theorem (Sankar '22+): $\delta_{\text{hom}}(C_5) > 0$.

The graph removal lemma

Theorem (Ruzsa-Szemerédi '78)

If G contains εn^2 edge-disjoint copies of H, then G contains $\delta n^{v(H)}$ copies of H, where $\delta = \delta_H(\varepsilon) > 0$.

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Question: Can we do better if G has high minimum degree?

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Note: $\delta_{\text{poly-rem}}(H) \leq \delta_{\text{lin-rem}}(H)$.

Theorem (Fox-Wigderson '21)

- *If $\delta(G) \geq (\frac{2r-5}{2r-2} + \alpha)n$ and G has εn^2 edge-disjoint copies of K_r, then G has $\Omega(\alpha\varepsilon n^r)$ copies of K_r.*
- *There are graphs G with $\delta(G) = (\frac{2r-5}{2r-2} - \alpha)n$ and εn^2 edge-disjoint copies of K_r, but only $\varepsilon^{\Omega(\log 1/\varepsilon)} n^r$ copies of K_r.*

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- Do $\delta_{\text{poly-rem}}(H)$, $\delta_{\text{lin-rem}}(H)$ receive finitely or infinitely many values on r-chromatic graphs H?
- Is there a relation between the removal thresholds $\delta_{\text{poly-rem}}(H)$, $\delta_{\text{lin-rem}}(H)$ and $\delta_{\chi}(H)$, $\delta_{\text{hom}}(H)$?

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Corollary

$\delta_{\textit{poly-rem}}(H)$ *receives infinitely many values on 3-chromatic* H.

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Theorem (Gishboliner, J., Sudakov)

If H is 3-chromatic,

$$\delta_{\textit{lin-rem}}(H) = \begin{cases} \frac{1}{2} & H \text{ has no critical edge}, \\ \frac{1}{3} & H \text{ has a critical edge and contains a triangle}, \\ \frac{1}{4} & H \text{ has a critical edge but no triangle}. \end{cases}$$

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Corollary

$\delta_{\textit{lin-rem}}(H)$ receives 3 different values on 3-chromatic H.

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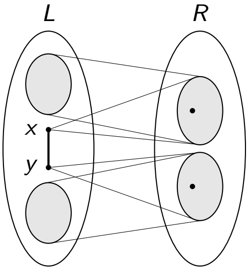

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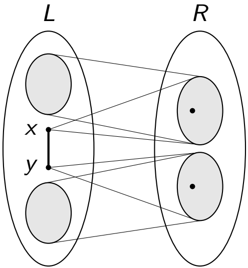

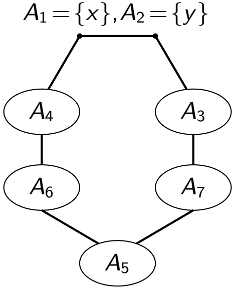

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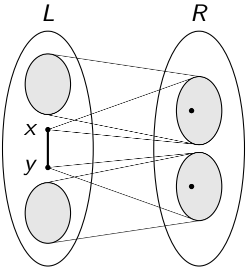

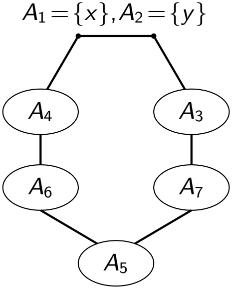

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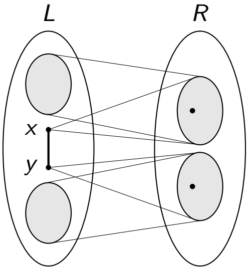

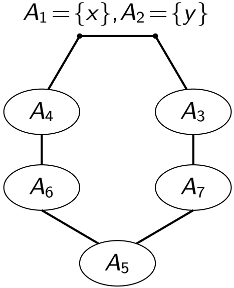

- $H \to C_{2k+1}$ for some $k \geq 2$ with $A_1 = \{x\}, A_2 = \{y\}$.
- Consider $H = C_5$.

Linear removal lemma threshold for C_5

Condition: $H = C_5$, $\delta(G) > (\frac{1}{4} + \alpha)n$, $> \varepsilon n^2$ edge-disjoint C_5-copies.

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An "ideal" proof for C_5

Condition: $H = C_5$, $\delta(G) > (\frac{1}{4} + \alpha)n$, $> \varepsilon n^2$ edge-disjoint C_5-copies, $< \varepsilon^c$ edge-disjoint copies of C_3 and C_5.

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  - "Ideally", G' is bipartite (large degree implies small girth).

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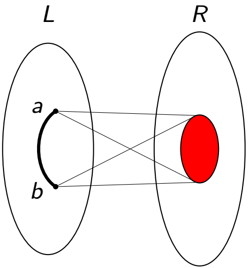

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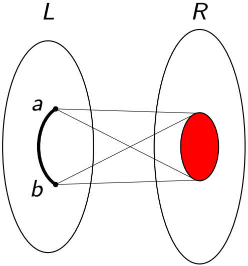

  - $\Omega_\alpha(n^3)$ paths (x_1, x_2, x_3) with x_1, x_3 red because $\delta(G) > \frac{1}{4}n$.
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An "ideal" proof for C_5

Condition: $H = C_5$, $\delta(G) > (\frac{1}{4} + \alpha)n$, $> \varepsilon n^2$ edge-disjoint C_5-copies in G, G' is bipartite with bipartition $L \sqcup R$.

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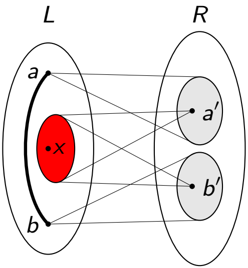

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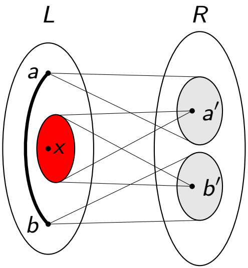

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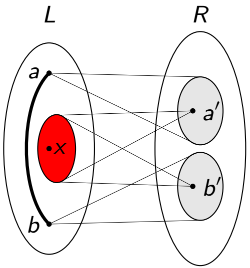

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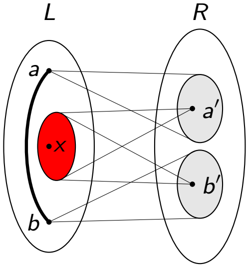

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- $\Omega_\alpha(n^5)$ C_5 in total.

Linear removal lemma threshold for C_5

Condition: $H = C_5$, $\delta(G) > (\frac{1}{4} + \alpha)n$, $> \varepsilon n^2$ edge-disjoint C_5-copies and $< \varepsilon^c$ edge-disjoint copies of C_3 and C_5.

- Let E_c be the set of edges in these C_3 and C_5. $|E_c| < \frac{\alpha^2}{100} n^2$.

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- Let $G' := G \setminus E_c \setminus S$.

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- Consider when G' is bipartite.

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Condition: $H = C_5$, $\delta(G') > (\frac{1}{4} + \frac{\alpha}{2})n$, G' is bipartite, $> \varepsilon n^2$ edge-disjoint C_5 in G.

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  - Or at least $\frac{\varepsilon n^2}{2}$ edges are of type II.

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Linear removal lemma threshold for C_5

Case 1: at least $\frac{\varepsilon n^2}{2}$ edges are of type I.

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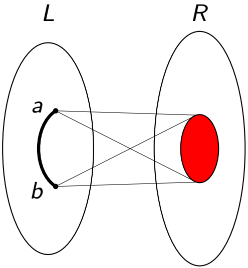

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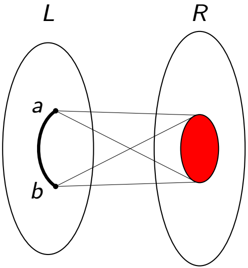

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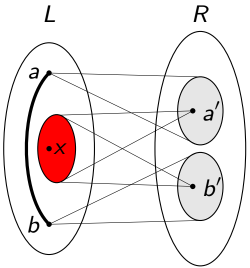

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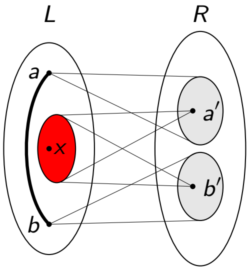

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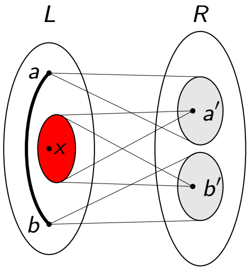

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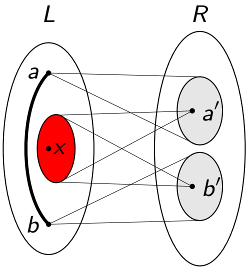

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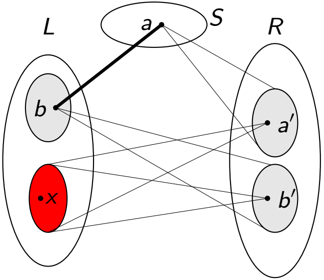

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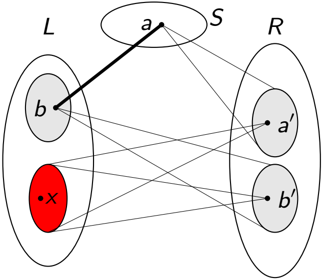

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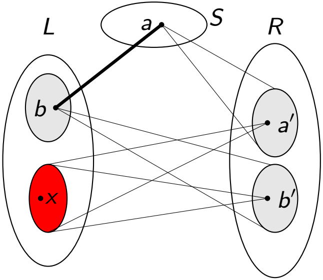

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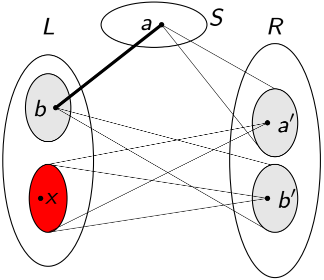

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Open questions

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- Is $\delta_{\text{poly-rem}}(H) = \delta_{\text{hom}}(\mathcal{I}_H)$?

The End

Questions? Comments?