Difference-isomorphic graph families

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Joint work with Lior Gishboliner and Benny Sudakov

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Set families with certain properties

Theorem (Sperner 1928)

If $\mathcal{A}$ *is a family of distinct subsets of* $[n]$ *s.t. no set is contained in the other, then* $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

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- If $|\mathcal{A}| \approx \binom{n-1}{r-1}$, then $\mathcal{A}$ is "close" to the extremal example.

Graph families with intersection properties

Conjecture (Simonovits-Sós 1976)

If $\mathcal{G}$ is a family of graphs on $[n]$ s.t. every two graphs in $\mathcal{G}$ share a common triangle, then $|\mathcal{G}| \leq 2^{\binom{n}{2}-3}$.

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- $\triangle \Rightarrow C_n$, $\quad |G| \leq 2^{\binom{n}{2}-n}$ by Leader, Ranđelović and Tan.

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Conjecture (Gowers, 2009)

$\forall \delta > 0\ \forall n \gg_\delta 1$, if $\mathcal{A}$ is a family of δ-proportion of matrices in $\{0, \ldots, k-1\}^{n \times n}$, then $\exists A_0, \ldots, A_{k-1} \in \mathcal{A}$ and $X \subseteq [n]$ s.t. $A_i[X \times X]$ is an all-i matrix, $i = 0, 1, \ldots, k-1$, and everywhere else the same.

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Question(Alon 2023+)

When $\mathcal{H}$ contains all the K_4's, does $|\mathcal{G}| = o(2^{\binom{n}{2}})$?

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- Does $G \in \mathcal{G}$ look *alike*?
  - If so, maybe $|\mathcal{G}| \leq n^{O(n)}$ as there are $n!$ isomorphisms.

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Setting: $n = 2k$, $U = \{u_1, \ldots, u_k\}$, $V = \{v_1, \ldots, v_k\}$, $\quad u_i \overset{\psi}{\longleftrightarrow} v_i$.

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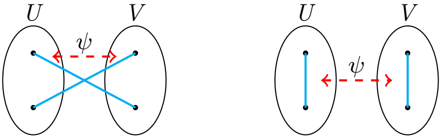

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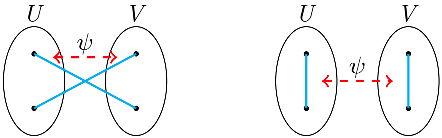

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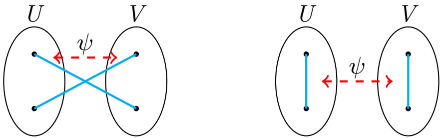

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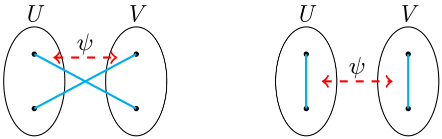

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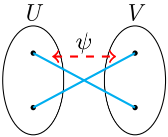

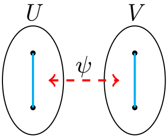

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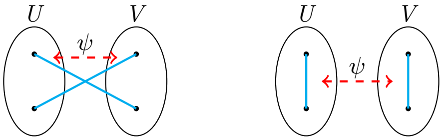

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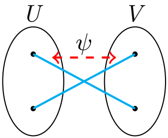

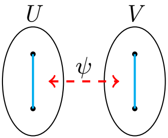

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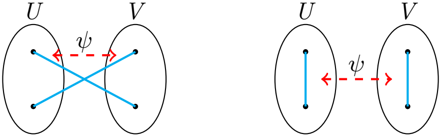

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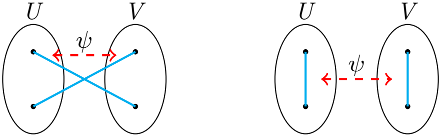

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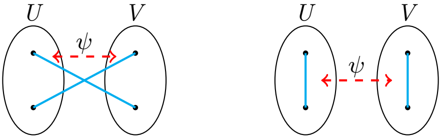

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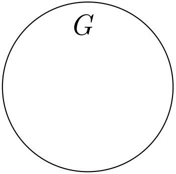

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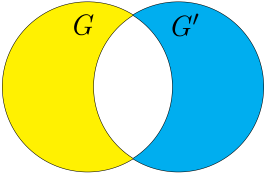

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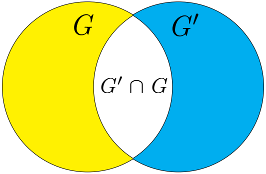

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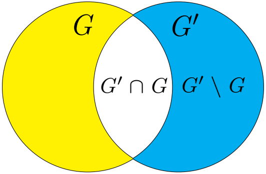

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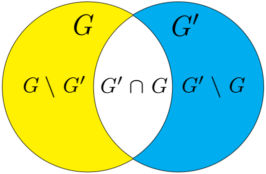

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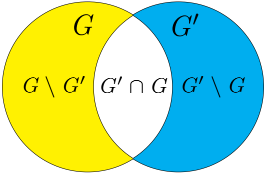

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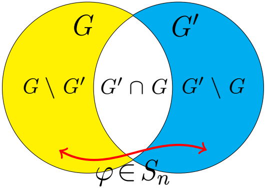

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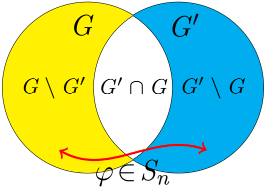

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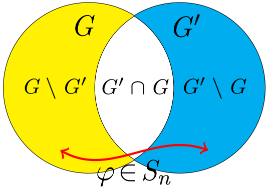

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A sketch of the upper bound proof

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  - In particular, ψ is an involution.
  - $\exists$ large ψ-clique in $N_\varphi(G) \Rightarrow (\beta)$.

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- Assume (α) happens; (β) is more complicated.

A sketch of the upper bound proof

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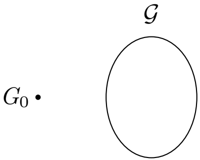

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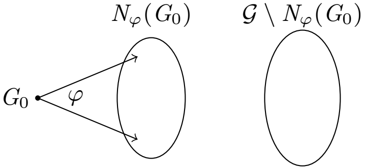

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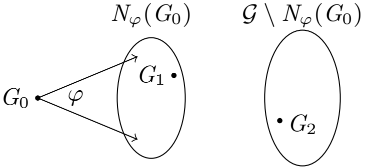

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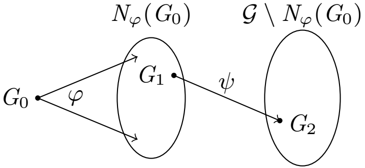

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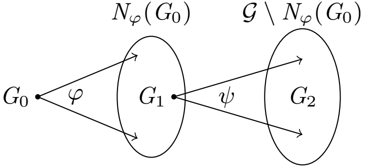

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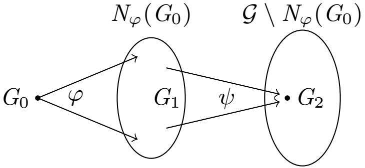

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Recall: $M = 2^{\frac{1}{2}(\binom{n}{2} - \frac{n}{2})}, |\mathcal{G}|/n! \leq |N_\varphi(G_0)| < M \cdot 2^{-n/100}$.

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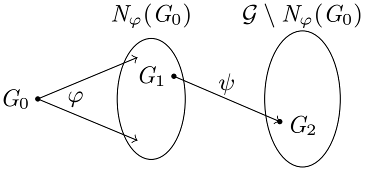

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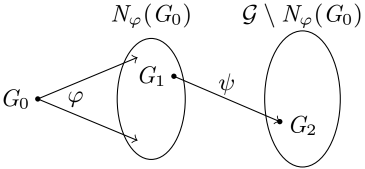

$$
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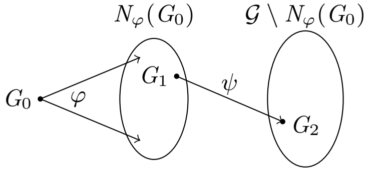

$$\begin{aligned} & |N_\varphi(G_0)||\mathcal{G} \setminus N_\varphi(G_0)| \\ \leq\ & 2^{o(n)} \cdot |N_\varphi(G_0)| \cdot M \cdot 2^{-n/200} + n! \cdot |\mathcal{G} \setminus N_\varphi(G_0)| \cdot M \cdot n^{-10n} \end{aligned}$$

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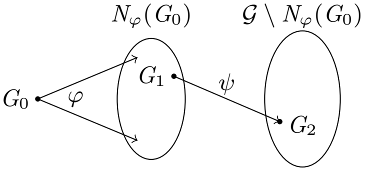

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$$\Rightarrow |\mathcal{G} \setminus N_\varphi(G_0)| \leq 2^{o(n)} \cdot M \cdot 2^{-n/200}$$

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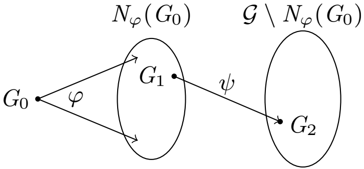

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\Rightarrow\ & |\mathcal{G} \setminus N_\varphi(G_0)| \le 2^{o(n)} \cdot M \cdot 2^{-n/200} \text{ or } |N_\varphi(G_0)| \le n! \cdot M \cdot n^{-10n}.
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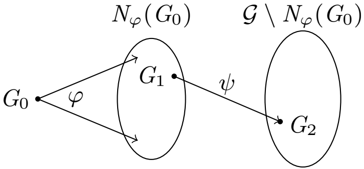

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$$\Rightarrow \quad |\mathcal{G}| \leq M \cdot 2^{-n/200} + 2^{o(n)} \cdot M \cdot 2^{-n/200}$$

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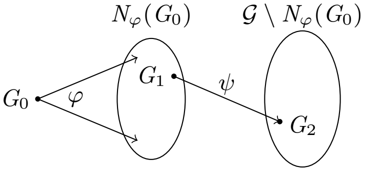

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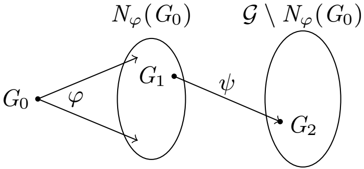

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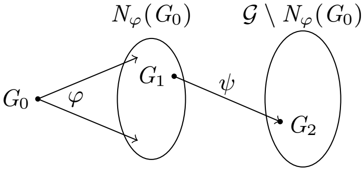

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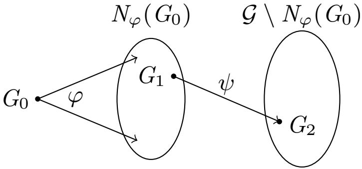
& \text{or } |\mathcal{G}| \le n!|N_\varphi(G_0)| \le (n!)^2 \cdot M \cdot n^{-10n}
\end{aligned}
$$

A sketch of the upper bound proof

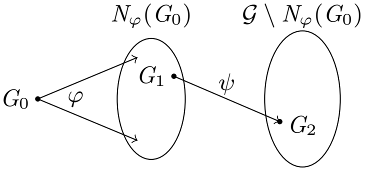
Recall: $M = 2^{\frac{1}{2}(\binom{n}{2} - \frac{n}{2})}, |\mathcal{G}|/n! \leq |N_\varphi(G_0)| < M \cdot 2^{-n/100}$.



$$
\begin{aligned}
& |N_\varphi(G_0)||\mathcal{G} \setminus N_\varphi(G_0)| \\
\leq\ & 2^{o(n)} \cdot |N_\varphi(G_0)| \cdot M \cdot 2^{-n/200} + n! \cdot |\mathcal{G} \setminus N_\varphi(G_0)| \cdot M \cdot n^{-10n} \\
\Rightarrow & |\mathcal{G} \setminus N_\varphi(G_0)| \leq 2^{o(n)} \cdot M \cdot 2^{-n/200} \text{ or } |N_\varphi(G_0)| \leq n! \cdot M \cdot n^{-10n}. \\
\Rightarrow \quad & |\mathcal{G}| \leq M \cdot 2^{-n/200} + 2^{o(n)} \cdot M \cdot 2^{-n/200} \leq M \cdot 2^{-n/400} \\
& \text{or } |\mathcal{G}| \leq n!|N_\varphi(G_0)| \leq (n!)^2 \cdot M \cdot n^{-10n} \ll M \cdot 2^{-n/400}.
\end{aligned}
$$

A sketch of the upper bound proof

Recall: $M = 2^{\frac{1}{2}(\binom{n}{2}-\frac{n}{2})}, |\mathcal{G}|/n! \le |N_\varphi(G_0)| < M \cdot 2^{-n/100}$.



$$\begin{aligned}
& |N_\varphi(G_0)||\mathcal{G} \setminus N_\varphi(G_0)| \\
\le\ & 2^{o(n)} \cdot |N_\varphi(G_0)| \cdot M \cdot 2^{-n/200} + n! \cdot |\mathcal{G} \setminus N_\varphi(G_0)| \cdot M \cdot n^{-10n}
\end{aligned}$$

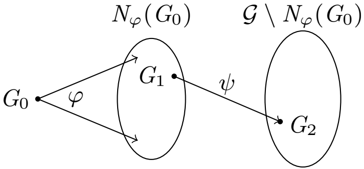
$$\Rightarrow |\mathcal{G} \setminus N_\varphi(G_0)| \le 2^{o(n)} \cdot M \cdot 2^{-n/200} \text{ or } |N_\varphi(G_0)| \le n! \cdot M \cdot n^{-10n}.$$

$$\begin{aligned}
\Rightarrow \quad & |\mathcal{G}| \le M \cdot 2^{-n/200} + 2^{o(n)} \cdot M \cdot 2^{-n/200} \le M \cdot 2^{-n/400} \\
& \text{or } |\mathcal{G}| \le n!|N_\varphi(G_0)| \le (n!)^2 \cdot M \cdot n^{-10n} \ll M \cdot 2^{-n/400}.
\end{aligned}$$

$$\Rightarrow |\mathcal{G}| \le M \cdot 2^{-n/400}!$$

A sketch of the upper bound proof

Recall: $M = 2^{\frac{1}{2}(\binom{n}{2} - \frac{n}{2})}, |\mathcal{G}|/n! \le |N_\varphi(G_0)| < M \cdot 2^{-n/100}$.



$$
\begin{aligned}
&|N_\varphi(G_0)||\mathcal{G} \setminus N_\varphi(G_0)| \\
&\le 2^{o(n)} \cdot |N_\varphi(G_0)| \cdot M \cdot 2^{-n/200} + n! \cdot |\mathcal{G} \setminus N_\varphi(G_0)| \cdot M \cdot n^{-10n} \\
\Rightarrow\ & |\mathcal{G} \setminus N_\varphi(G_0)| \le 2^{o(n)} \cdot M \cdot 2^{-n/200} \text{ or } |N_\varphi(G_0)| \le n! \cdot M \cdot n^{-10n}. \\
\Rightarrow\ & |\mathcal{G}| \le M \cdot 2^{-n/200} + 2^{o(n)} \cdot M \cdot 2^{-n/200} \le M \cdot 2^{-n/400} \\
& \text{or } |\mathcal{G}| \le n!|N_\varphi(G_0)| \le (n!)^2 \cdot M \cdot n^{-10n} \ll M \cdot 2^{-n/400}.
\end{aligned}
$$

$\Rightarrow |\mathcal{G}| \le M \cdot 2^{-n/400}$! *Nice!*

Further directions

Any suggestions?

The End

Questions? Comments?