Semi-algebraic and Semi-linear Ramsey Numbers

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Ramsey Numbers for Graphs

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Theorem (Erdős and Szekeres '35, Erdős '47)

$R(n,n) = 2^{\Theta(n)}$, $R(s,n) = n^{\Theta(s)}$.

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- **The exponential gap remains till now!**

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$R_r(s,n) \approx 2^{R_{r-1}(s,n)}$ *when* $r \geq 4$.

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Corollary (Erdős and Rado '52, Erdős, Hajnal and Rado '65)

$\mathrm{tw}_{r-1}(\Omega(n^2)) < R_r(n,n) < \mathrm{tw}_r(\mathcal{O}(n))$, *where* $\mathrm{tw}_r(n) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{r \text{ times}}$.

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- The **critical case** is when $r = 3$, i.e. 3-uniform hypergraphs.

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- **Sign-pattern**: $\text{sign}(f_1(p)), \dots, \text{sign}(f_m(p)) \in \{0, +, -\}^m$.
- Intersection graphs of certain geometric objects.

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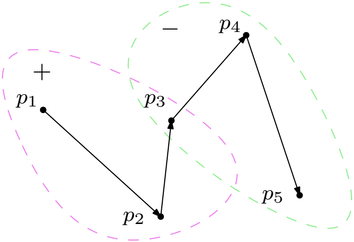

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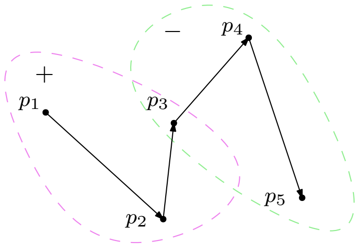

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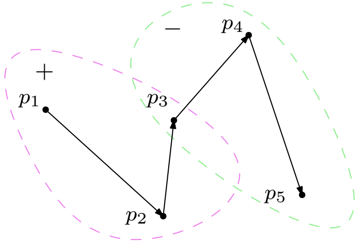

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- It is of complexity $(2, 2, 1)$.

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- In constrast, $R_3(n,n) = 2^{2^{\mathcal{O}(n)}}$.

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- We refuted this conjecture.

*Theorem (**J.**, Tomon '23)*

$R_3^{(d,D,m)}(4,n) > n^{\log^{1/3-o(1)}(n)} = 2^{\log^{1.3}(n)}$.

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$R_r^{(1,1,1)}(n,n) > 2^{\Omega(n^{r/2-1})}$ *for even r's.*

Open problems

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Is it true that $R_r^{(d,2,m)}(n,n) < \mathrm{tw}_k(n^{\mathcal{O}(1)})$ for absolute constant k?

The End

Questions? Comments?